CH # 2

Numerical Sequences & Series

Intuitive (of ideas) obtained by feelings rather than by considering facts:

Eventually # adv at the end of a period of time or a series of events or beyond some point of time.

Primitive # belonging to an early stages in the development.

ϵ – Neighbourhood

In any metric space (X, d) and $x_0 \in X$, open ball $B(x_0; \epsilon)$ is called an ϵ-neighbourhood (or sometimes neighbourhood)

If $x_0 \in R$, then the open interval $(x_0 - \epsilon, x_0 + \epsilon)$ is called an ϵ-neighbourhood of x_0. We denote neighbourhood by $N(x_0 : \epsilon)$ or $N_\epsilon(x_0)$.

Deleted Neighbourhood

The set $(x_0 - \epsilon, x_0) \cup (x_0, x_0 + \epsilon)$ is called a deleted neighbourhood of x_0 because x_0

deleted from $N_\epsilon(x_0)$. It is denoted by $N'_\epsilon(x_0)$

Sequence

A sequence in a set S is a function whose domain is the set N of natural numbers and range is a subset of the set S.

Real Sequence

A sequence of real numbers (or a sequence in R) is a function whose domain is the set N of natural numbers and range is a subset of the set R of real numbers.

Notations

(1) # If $x : N \longrightarrow R$ is a sequence, we usually denote the value of x at n by symbol x_n instead of the function notation $x(n)$. $x_1, x_2, \ldots x_n, \ldots$ are called terms of sequence

(2). The sequence $x : N \longrightarrow R$ is denoted by $\{x_n\}_{n=1}^{\infty}$ or $\{x_n\}$ or $\langle x_n \rangle$ or $(x_n : n \in N)$ or (x_n).

(3) Notation $\langle x_n \rangle$ or $(x_n : n \in N)$ or $\{(n, f(n)) : n \in N\}$ is used to that terms

...e sequence ... an ordering

...duced by the ... of natural n...

The set o... distinct terms ... called the r... and is denoted ... $x_n : n \in N\}$ a... ues in the ran... ...t ordered.

Since in a ... nce $\{x_n\}, n \in$ ... s an infinite ... therefore the n... terms of a sequ... is always infin... range of sequen... ay be a finite ... .$\}$

if $x_n = (-1)$... n $\{x_n\} = \{-1, 1,$ ... nite.

The range of ... ence $= \{-1, 1\}$ w... ng

...) # Sequences ... often defined ... a formula for ... nth term x_n ...

7)) # Sometimes ... s convenient to ... hen ... terms of a seq... e in order st... the rule of ... ation is seem... n

8)) # The m... nd nth term ... $x_m = x_n$ for $m \neq n$ ... reated distinc... sitions ie the terms ... curring at dif... ough are treated ... distinct term... they have ... e value.

Constant Sequence

A sequence $\{x_n\}$ defined by $x_n = c \in R$ $\forall n \in N$ is called a constant sequence.

OR

A sequence whose range is a singleton is called a constant sequence.

Mathematical Formula Notation For Sequence

Many sequences may be defined by some mathematical rule by two ways.

(a) By an explicit formula. (b) by a recusion or inductive or iterative formula.

(a) Explicit Formula

A sequence may be defined by giving an explicit formula for the nth term. e.g

(1) $a_n = \frac{1}{n}$ (2) $a_n = \frac{n}{n+1}$

(3) $a_n = (-1)\frac{n}{n+1}$ $a_n = 3$

(b) Recursive Formula

Sometimes sequences are defined by specifying (clearly giving) one or more initial terms and by giving a formula that relates each subsequent (next coming) term to the previous terms. Such

Sequences are said to be defined recursively or inductively or iteratively and the defining formula is called a recursion formula or inductive formula.

OR

A sequence defined by a formula for the nth term in terms of one or several previous terms with some initial terms specified clearly.

Examples #

$\boxed{1}$ $a_{n+2} = a_{n+1} + a_n$ $\quad a_1 = 1 \quad a_2 = 1$

1, 1, 2, 3, 5, 8 - - - .

2) # $a_n = \frac{a_{n+1}}{2}$ $\quad n \geq 1 \quad a_0 = 1$

3) # $a_n = n \cdot a_{n-1}$ $\quad a_1 = 1$

4) # Fibonacci Sequence.

$a_{n+1} = a_n + a_{n-1}$ $\quad a_1 = 1, \; a_2 = 1$

1, 1, 2, 3, 5 8, - - - .

Thus each term after the 1st two terms is the sum of its two immediate previous terms.

5) The sequence $\{2n\}$ of even numbers can be defined by $x_1 = 2 \quad x_{n+1} = x_n + 2$.

OR by $y_1 = 2 \quad y_{n+1} = y_1 + y_n$.

Note not every sequence has a simple formula or even any formula at all. e.g $\{d_n\}$, where d_n is nth digit in the decimal representation of π

Bounded Above Sequence.

A sequence $\{x_n\}$ is said to be bounded above if $\exists$ a real no K such that

$$x_n \leq K \quad \forall n \in N$$

i.e if the range of the sequence is bounded above.

Bounded Below Sequence.

A sequence $\{x_n\}$ is said to be bounded below if $\exists$ a real no k such that

$$x_n \geq k \quad \forall n \in N.$$

i.e if the range of the sequence is bounded below.

Bounded Sequence

A sequence is said to be bounded if it is bounded above as well as below:
Thus a sequence $\{x_n\}$ is bounded if $\exists$ two numbers k & K ($k \leq K$) such that

$$k \leq x_n \leq K \quad \forall n \in N.$$

i.e if the range set $\{x_n : n \in N\}$ is bounded.

A sequence that is not bounded is said to be unbounded sequence.

Unbounded Above

A sequence $\{x_n\}$ is said to be unbounded above if it is not bounded above i.e if for every real no K $\exists$ $m \in N$ s. that

$$a_m > K.$$

Unbounded Below

A sequence $\{x_n\}$ is said to be unbounded below if it is not bounded below i.e. if for every real no k $\exists$ $m \in N$ s.t

$$a_m < k$$

Examples

(1) # The sequence $\{a_n\}$ defined by $a_n = \frac{1}{n}$ is bounded because. $0 < a_n \leq 1$.

(2) # The sequence $\{a_n\}$ defined by $a_n = n$ is bounded below by 1 because $a_n \geq 1$ $\forall n \geq N$. It is not bounded above because $\exists$ no real no K such that

$$a_n \leq K \quad \forall n \in N$$

(3)# The sequence $\{(-1)^n\}$ is bounded because.

$-1 \le a_n \le 1 \quad \forall n \in N$

(4)# The sequence $\{-n\}$ is bounded above ~~below~~ because

$a_n \le -1 \quad \forall n \in N$ & is not bounded below

(5)# Every Constant sequence is bounded.

(6)# The sequence $\{a_n\}$ defined by $a_n = (-1)^n \cdot n$ is neither bounded above nor bounded below.

Theorem# A sequence $\{a_n\}$ is bounded iff $\exists$ a +ve real no M such that

$$|a_n| \le M \quad \forall n \in N.$$

Proof# Necessary Condition

Let $\{a_n\}$ be bounded. The $\exists$ two real numbers h & k such that

$$h \le a_n \le k \quad \forall n \in N.$$

Let $M = \text{Max}\{|h|, |k|\}$. Then.

$|h| \le M$ & $|k| \le M.$

$\Rightarrow -M \le h \le M$ & $-M \le k \le M.$

$\Rightarrow -M \le h \le a_n \le k \le M \quad \forall n \in N$

$\Rightarrow -M \le a_n \le M \quad \forall n \in N$

$\Rightarrow |a_n| \le M \quad \forall n \in N.$

Sufficient Condition #

Let M be +ve real no such that

$|a_n| \leq M \quad \forall n \in N$.

Then $-M \leq a_n \leq M \quad \forall n \in N$.

$\Rightarrow \{a_n\}$ is bounded.

Note The above Theorem is used as definition of bounded sequence.

Theorem # If $\{a_n\}$ & $\{b_n\}$ are bounded sequences and c is a real no, then

(a) $\{a_n + b_n\}$ is bounded.

(b) $\{a_n b_n\}$ is bounded.

(c) $\{c a_n\}$ is bounded.

Proof # $\because$ $\{a_n\}$ & $\{b_n\}$ are bounded.

$\therefore$ $\exists$ +ve numbers M_1 & M_2

Such that $|a_n| \leq M_1$ & $|b_n| \leq M_2 \quad \forall n \in N$.

$\Rightarrow |a_n| + |b_n| \leq M_1 + M_2 = M \quad \forall n$

Now $|a_n + b_n| \leq |a_n| + |b_n| \leq M \quad \forall n \in N$.

$\Rightarrow \{a_n + b_n\}$ is bounded.

(b) $|a_n b_n| = |a_n||b_n| \leq M_1 M_2 = M_3 \quad \forall n$

$\Rightarrow \{a_n b_n\}$ is bounded.

(c) # $|Ca_n| = |c||a_n| = K/M_1 = M_1 \quad \forall n$

$\frac{8}{10}$

$\Rightarrow \{ca_n\}$ is bounded.

Eventual Property of a Sequence.

If the terms of a sequence do not have a certain property from start but have that property from some point (from some term) on, then the sequence has that property eventually.

If a sequence $\{a_n\}$ fulfil ~~some~~ property P eventually, then mathematically we say that $\exists$ an integer $n_1 \in N$ such that if $n \geq n_1$, $\{a_n\}$ satisfies property P.

Limit of a Sequence & Convergence

A sequence $\{a_n\}$ in R is said to converge to $l \in R$ or l is a limit of $\{a_n\}$ if for every $\epsilon > 0$ $\exists$ a natural no $n_1(\epsilon)$ such that

$$|a_n - l| < \epsilon \qquad \forall n \geq n_1$$

If so we write $\lim_{n\to\infty} a_n = l$ or $\lim a_n = l$. or $a_n \to l$.

If a sequence has a limit, then sequence is convergent, if it has no limit, the sequence is dgt.

Note (1) A real no l is a limit of sequence $\{a_n\}$ if given $\epsilon > 0$, all but a finite no of terms of $\{a_n\}$ lie within ϵ of l

$\forall n \geq n_1$

(2) $|a_n - l| < \epsilon \quad \forall n \geq n_1$

$\Rightarrow l - \epsilon < a_n < l + \epsilon \quad \forall n \geq n_1$

$\Rightarrow a_n \in (l - \epsilon, l + \epsilon) \quad \forall n \geq n_1$

$\Rightarrow$ given any $\epsilon > 0$, all the terms of the sequence, except the 1st $n_1 - 1$ terms lie in the interval $(l - \epsilon, l + \epsilon)$. The 1st $n_1 - 1$ terms may be scattered anywhere. The number $n_1 - 1$ of terms left out of the interval $(l - \epsilon, l + \epsilon)$ depends upon the size of ϵ. The smaller the size of ϵ, the larger will be the no of terms left out of $(l - \epsilon, l + \epsilon)$.

(2) We say that a sequence $\{a_n\}$ ~~has~~ ultimately has a certain property if $\exists$ a no n_1 such that sequence $\{a_n\}$ satisfies that property for $n \geq n_1$. A sequence $\{a_n\}$ converges to l if the terms of $\{a_n\}$ are ultimately in every ϵ-neighbourhood of l.

(3) **With the Language of Nbhd.**

A sequence $\{a_n\}$ converges to number l if for each ϵ- nbhd $N_\epsilon(l)$ of l all but a finite no of terms of $\{a_n\}$ belong to $N_\epsilon(l)$

Examples

(a) $$\lim_{n\to\infty} (1/n) = 0$$

Let $a_n = \frac{1}{n}$ and $\epsilon > 0$ be given.

$$|a_n - l| = |\frac{1}{n} - 0| = \frac{1}{n}.$$

and $|a_n - l| < \epsilon$ if $\frac{1}{n} < \epsilon$.

if $n > \frac{1}{\epsilon}$.

Thus if we take natural no n_1 greater than real no $\frac{1}{\epsilon}$, then we have.

$$|a_n - l| < \epsilon \quad \forall\, n \geq n_1$$

$\Rightarrow a_n \longrightarrow 0$

Explaination For each $\epsilon > 0$, we can always trace n_1 by relation $n > \frac{1}{\epsilon}$ let us see how.

13

$\underline{\underline{\epsilon}}$

for $\epsilon = .1$

$\frac{1}{\epsilon} = \frac{1}{.1} = 10$

$\underline{\underline{n_1}}$

our n_1 will be greater than 10 because for $a_{10} = \frac{1}{10}$ we have

$|a_{10} - 0| = |\frac{1}{10} - 0| = .1 \nless$.

$a_{11} = \frac{1}{11}$

$|a_{11} - 0| = \frac{1}{11} = .09 < \epsilon = .1$

Thus for $\epsilon = .1$ $n_1 = 11$ &

$|a_n - l| < \epsilon \quad \forall n \geq n_1$

for $\epsilon = .01$

$\frac{1}{\epsilon} = \frac{1}{.01} = 100$

n_1 will be 101 or greater and

$|a_n - l| < \epsilon = .01 \quad \forall n \geq 101 = n_1$

We note that for smaller ϵ, the greater n_1

For $\epsilon = .5$

$\frac{1}{\epsilon} = \frac{1}{.5} = 2$

n_1 will be 3 or greater and

$|a_n - l| < \epsilon = .5 \quad \forall n \geq 3 = n_1$

We note that for greater ϵ, the smaller n_1

(b) $\lim \frac{1}{n^2+1} = 0$

let $a_n = \frac{1}{n^2+1}$ $\quad l = 0 \quad$ let $\epsilon > 0$ be given

$|a_n - l| = |\frac{1}{n^2+1} - 0| = \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n}$

Thus if $\frac{1}{n} < \epsilon$ or $n > \frac{1}{\epsilon}$

we have for n_1 greater than $\frac{1}{\epsilon}$

$$|a_n - l| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow a_n \longrightarrow 0$$

(c) $$\lim_{n\to\infty} \left(\frac{3n+2}{n+1}\right) = 3$$

Let $a_n = \frac{3n+2}{n+1}$ $\quad l = 3$ & $\epsilon > 0$ be given

$$|a_n - l| = \left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{3n+2-3n-3}{n+1}\right|$$

$$= \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}$$

Thus $|a_n - l| < \epsilon$ $\quad$ if $\frac{1}{n} < \epsilon$

So we can take $\quad$ if $n > \frac{1}{\epsilon}$

for each $\epsilon > 0$ n_1 greater than $\frac{1}{\epsilon}$ such that

$$|a_n - l| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow a_n \longrightarrow l = 3$$

(d) $$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$$

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$|a_n - l| = \left|\frac{1}{\sqrt{n+1}+\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}+\sqrt{n}}$$

$$= \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}}$$

if $\frac{1}{\sqrt{n}} < \epsilon$, Then $n > \frac{1}{\sqrt{\epsilon}}$

Thus for each $\epsilon > 0$, if we take n_1 greater than $\frac{1}{\sqrt{\epsilon}}$, we have.

$$|a_n - l| < \epsilon \qquad \forall n \geq n_1$$

$$\Rightarrow a_n \longrightarrow l = 0$$

(e) $\lim \left(\frac{n^2-1}{2n^2+3} \right) = \frac{1}{2}.$

$a_n = \frac{n^2-1}{2n^2+3}$ $\qquad l = 1/2 \qquad$ let $\epsilon > 0$

$$|a_n - l| = \left| \frac{n^2-1}{2n^2+3} - \frac{1}{2} \right|$$

$$= \left| \frac{2n^2-2-2n^2-3}{2(2n^2+3)} \right| = \frac{5}{4n^2+6} < \frac{5}{4n^2} < \frac{5}{n^2}$$

$|a_n - l| < \epsilon \qquad$ if $\frac{5}{n^2} < \epsilon$

if $5/\epsilon < n^2$

if $n > \sqrt{\frac{\epsilon}{5}}$

$\Rightarrow$ If we take n_1 greater than $\sqrt{\frac{\epsilon}{5}}$; then

$$|a_n - l| < \epsilon \qquad \forall n \geq n_1$$

$$\Rightarrow a_n \longrightarrow l = 1/2.$$

(f) If $0 < b < 1$, then $\lim b^n = 0$

for any $a > 0$, we can write.

$$b = \frac{1}{1+a} < 1.$$

$$\Rightarrow a = \frac{1}{b} - 1 > 0 \qquad \because b < 1$$

$a+1 = \frac{1}{b}$

Now By Bernoulli's Inequality

$$(1+a)^n \geqslant 1+na$$

Hence.

$$0 < b^n = \frac{1}{(a+1)^n} \leq \frac{1}{1+na} < \frac{1}{na}.$$

$$|a_n - 0| = |b^n - 0| = b^n < \frac{1}{na}.$$

$|a_n - 0| < \epsilon$ if $\frac{1}{na} < \epsilon$

if $\frac{1}{n} < a\epsilon$.

if $n > \frac{1}{a\epsilon}$.

$\Rightarrow$ we can take $n_1 > \frac{1}{a\epsilon}$ for each $\epsilon > 0$

Such that

$|a_n - 0| < \epsilon \quad \forall n \geqslant n_1$

<u>OR</u>

$|a_n - 0| = b^n$

$|a_n - 0| < \epsilon$ if $b^n < \epsilon$

if $n \ln b < \ln \epsilon$

if $n > \frac{\ln \epsilon}{\ln b}$ $\because \ln b < 0$

Thus if we choose no $n_1 > \frac{\ln \epsilon}{\ln b}$, we have

$|a_n - 0| < \epsilon \quad \forall n \geqslant n_1$

e.g if $b = .8$ & if $\epsilon = .01$ we have $n_1 > \frac{\ln .01}{\ln .8}$

≈ 20.6377. Thus $n_1 = 21$ would be appropriate for $\epsilon = .01$

17

Uniquness of Limit

Theorem# A cgt. sequence in R can have at most one limit

OR

Every Convergent sequence in R has a unique limit

OR

The limit of the sequence, if it exists, is unique.

Proof# Let $\{a_n\}$ be an arbitrary Convergent sequence and A, B be two limits of the sequence.

Suppose that $A \neq B$

Now $\frac{|A-B|}{2} > 0$

$\because$ $\{a_n\}$ Converges to A

$\therefore$ for $\epsilon = \frac{|A-B|}{2}$ $\exists$ a natural no n_1 such that

$$|a_n - A| < \frac{|A-B|}{2} \longrightarrow (1) \quad \forall n \geq n_1$$

$\because$ $\{a_n\}$ Converges to B

$\therefore$ for $\epsilon = \frac{|A-B|}{2}$ $\exists$ a natural no n_2. Such that

$$|a_n - B| < \frac{|A-B|}{2} \quad \forall n \geq n_2 \longrightarrow (2)$$

Let $n_3 = \text{Max}\{n_1, n_2\}$

Then

$|a_n - A| < \frac{|A-B|}{2}$

and

$|a_n - B| < \frac{|A-B|}{2} \quad \forall n \geq n_3 \rightarrow (3)$

$\forall n \geq n_3 \rightarrow (4)$

$$|A-B| = |A - a_n + a_n - B|$$
$$\leq |A - a_n| + |a_n - B|$$
$$< \frac{|A-B|}{2} + \frac{|A-B|}{2} \quad \forall n \geq n_3$$

$\Rightarrow |A-B| < |A-B|$

which is absurd. Hence $A = B$

$\Rightarrow$ limit is unique.

OR

$\because \lim_{n\to\infty} a_n = A \qquad \lim_{n\to\infty} a_n = B$

$\therefore$ For any $\epsilon > 0$ $\exists$ n_1 & n_2 such that

$|a_n - A| < \epsilon/2 \quad \forall n \geq n_1$

$|a_n - B| < \epsilon/2 \quad \forall n \geq n_2$

Let $n_3 = \max(n_1, n_2)$

Then

$|a_n - A| < \epsilon/2. \quad \forall n \geq n_3$

$|a_n - B| < \epsilon/2 \quad \forall n \geq n_3$

$|A - B| = |A - a_n + a_n - B|$

$$|A-B| \leq |A-a_n| + |a_n - B|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \forall n \geq n_3$$
$$\Rightarrow |A-B| < \epsilon \quad \forall n \geq n_3$$

$\because$ ϵ is an arbitrary +ve quantity

$\therefore$ $|A-B|$ is less than every +ve quantity however small and so must be zero

Thus $|A-B| = 0$

$\Rightarrow |A-B| = 0$

$\Rightarrow A-B = 0 \Rightarrow A = B$

$\Rightarrow$ Limit of sequence is unique.

<u>Theorem</u> # Let $\{x_n\}$ be a sequence of real numbers and let $x \in R$. The following statements are equivalent

(a) # $\{x_n\}$ converges to x.

(b) # For every $\epsilon > 0$ $\exists$ a natural no K. such that

$$|x_n - x| < \epsilon \quad \forall n \geq K.$$

(c) # For every $\epsilon > 0$, $\exists$ a natural no. K. s.that

$$x - \epsilon < x_n < x + \epsilon \quad \forall n \geq K.$$

(d) # For every neighbourhood $V_\epsilon(x)$ of x $\exists$ a natural no K such that $\forall n \geq k$, $x_n \in V_\epsilon(x)$

a (a) $\Longrightarrow$ b $\Rightarrow$ c $\Rightarrow$ d

Let $\{x_n\}$ be convergent to x.
Then by definition for every $\epsilon > 0$ $\exists$ a natural no K such that

$$|x_n - x| < \epsilon \quad \forall n \geq K.$$

$\Rightarrow$ $x - \epsilon < x_n < x + \epsilon \quad \forall n \geq K.$

$\Rightarrow$ $x_n \in (x - \epsilon, x + \epsilon) \quad \forall n \geq K.$

$\Rightarrow$ for every ϵ-nbhd $V_\epsilon(x)$ $\exists$ a natural no K such that

$$x_n \in V_\epsilon(x) \quad \forall n \geq K.$$

Divergent Sequence

(a) A sequence $\{a_n\}$ is said to diverge to $+\infty$ if given any +ve real no K, howerver, large, $\exists$ a natural no n_1 such that

$$a_n > K \quad \forall n \geq n_1$$

and we write

$$\lim_{n \to \infty} a_n = \infty \text{ or } a_n \to \infty \text{ as } n \to \infty$$

(b) A sequence $\{a_n\}$ is said to diverge to $-\infty$ if given any +ve real no K,

however large, $\exists$ a natural no n_1.
Such that

$$a_n < -K \quad \forall n \geq n_1$$

We write $\lim_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$

Equivalently a sequence $\{a_n\}$ diverges to $-\infty$ if given any $-ve$ real no k $\exists$ a natural no n_1 s.t

$$a_n < k \quad \forall n \geq n_1$$

(c) A sequence $\{a_n\}$ is said to be divergent sequence if it diverges to $+\infty$ or $-\infty$

Examples

(i) The sequences $\{n\}$ & $\{n^2\}$ diverge to ∞

(ii) The sequence $\{-n\}$ & $\{-n^2\}$ diverge to $-\infty$

Oscillatory Sequence

If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence.

Oscillatory sequences are of Two types

(a) A bounded sequence which does not Converge is said to oscillate finitely.

(b) e.g $\{(-1)^n\}$

Here $a_n = (-1)^n$ $\quad a_{2n} = (-1)^{2n} = 1$

$a_{2n+1} = (-1)^{2n+1} = -1$

$\lim_{n\to\infty} a_{2n} = 1 \qquad \lim_{n\to\infty} a_{2n+1} = -1$

$\Rightarrow$ Lim a_n does not exist $\Rightarrow$ sequence does not Converge. It is bounded because

$|a_n| = 1$

Hence sequence oscillate finitely

(b) An unbounded sequence which does not diverge is said to oscillate infinitely. e.g $\{(-1)^n n\}$

$a_n = (-1)^n n$

$\lim_{n\to\infty} a_{2n} = \lim (2n) = \infty$

$\lim a_{2n+1} = \lim (-1)^{2n+1}(2n+1) = -\infty$

Thus sequence does not diverge

Hence this sequence oscillates infinitely

Null Sequence

A sequence which converges to zero is said to be a null sequence.

e.g $\{\frac{1}{n}\}$, $\{\frac{1}{n^2}\}$, $\{\frac{1}{2n}\}$ & $\{\frac{(-1)^{n-1}}{n}\}$

are null sequences

Note A sequence $\{a_n\}$ is called infinitely small if $\lim a_n = 0$ & infinitely large if $\lim a_n = \infty$

Theorem # Let $\{x_n\}$ be a sequence of real numbers and $x \in R$. If $\{a_n\}$ is a sequence of +ve real nos. with $\lim a_n = 0$ and if for some $c > 0$ and some natural no n_1, we have

$$|x_n - x| \leq c a_n \quad \forall n \geq n_1$$

Then it follows that $\lim\limits_{n\to\infty} (x_n) = x$

Proof $\because \lim\limits_{n\to\infty} a_n = 0$

$\therefore$ For given $\epsilon > 0$ $\exists$ exists a natural no $k(\epsilon/c)$ such that

$$|a_n - 0| < \epsilon/c \quad \forall n \geq k.$$

$$\Rightarrow a_n < \epsilon/c \quad \forall n \geq k.$$

Let $n_2 = \text{Max}(K, n_1)$

Then

$$|x_n - x| \le c a_n \quad \forall n \geq n_2 \longrightarrow (1)$$

and

$$a_n < \epsilon/c \quad \forall n \geq n_2 \longrightarrow (2)$$

By (1) and (2)

$$|x_n - x| \le c a_n < c(\epsilon/c) = \epsilon \quad \forall n \geq n_2.$$

Since ϵ is arbitrary, we have

$$\lim_{n\to\infty} x_n = x$$

Examples

1)# If $a > 0$, then $\lim \left(\frac{1}{1+na}\right) = 0$

Sol # $\because$ $a > 0$

$\therefore$ $0 < na < 1 + na$.

$$\Rightarrow 0 < \frac{1}{1+na} < \frac{1}{na} = \frac{1}{a}\left(\frac{1}{n}\right)$$

Thus

$$\left|\frac{1}{1+na} - 0\right| \le \left(\frac{1}{a}\right)\frac{1}{n} \quad \forall n \in N.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{a} > 0$ & $n_1 = 1$

from above theorem we have.

$$\lim_{n\to\infty}\left(\frac{1}{1+na}\right) = 0$$

2)# if $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0$

Sol $\because$ $0 < b < 1$

$\therefore \quad b = \frac{1}{1+a} \Rightarrow a = \frac{1}{b} - 1$

So $a > 0 \quad (\because b < 1)$

By Bernoulli's Inequality, we have

$$(1+a)^n \geq 1+na$$

Hence

$$0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}$$

$\Rightarrow \quad b^n < \left(\frac{1}{a}\right)\frac{1}{n} \qquad \forall n \geq 1.$

$\because \lim_{n\to\infty} \frac{1}{n} = 0 \quad \& \quad \frac{1}{a} = c > 0$

$\therefore$ By above theorem

$\lim_{n\to\infty} b^n = 0$

3) If $c > 0$, then $\lim_{n\to\infty} c^{1/n} = 1$

Case 1 If $c = 1$, then sequence $\{c^{1/n}\}$ is constant sequence $\{1, 1, 1, \dots\}$ which converges to 1.

Case II If $c > 1$, then $c^{1/n} = 1 + d_n$, some $d_n > 0$

By Bernoulli's Inequality

$c = (1+d_n)^n \geq 1 + nd_n \quad \forall n \in N.$

$\Rightarrow c - 1 \geq nd_n$

$\Rightarrow d_n \leq \frac{c-1}{n} \qquad \forall n \in N.$

Now $|c^{1/n} - 1| = d_n \leq (c-1)\frac{1}{n} \quad \forall n \quad \{c - 1 > 0 \ \because c > 1$

By above theorem.

$$\lim_{n\to\infty} c^{1/n} = 1$$

Case III If $0 < c < 1$, then $c^{1/n} = \frac{1}{1+h_n}$ for some $h_n > 0$. By Bernoullis Inequality we have

$$c = \frac{1}{(1+h_n)^n} \le \frac{1}{1+nh_n} < \frac{1}{nh_n}$$

$$\Rightarrow \quad 0 < h_n < \frac{1}{nc} \quad \forall n \in N.$$

$$0 < 1 - c^{1/n} = \frac{h_n}{1+h_n} < h_n < \frac{1}{nc}.$$

$$\Rightarrow \quad |c^{1/n} - 1| < \left(\frac{1}{c}\right)\frac{1}{n} \quad \forall n \in N.$$

By above theorem

$$\lim_{n\to\infty} c^{1/n} = 1$$

4) $$\lim_{n\to\infty} n^{1/n} = 1$$

Sol $\because \; n^{1/n} > 1 \quad \forall n > 1$

Let $n^{1/n} = 1 + k_n$ for some $k_n > 0$ $\forall n > 1$

$$\Rightarrow n = (1+k_n)^n \quad \forall n > 1$$

$$= 1 + nk_n + \frac{1}{2}n(n-1)k_n^2 + \cdots \geq \frac{1}{2}n(n-1)k_n^2$$

$$\Rightarrow n \geq \frac{n(n-1)}{2}k_n^2 + 1$$

$\ldots 1 \geq \frac{n(n-1)}{2} h_n^2 \qquad \forall n > 1$

$h_n^2 \leq 2/n \qquad \forall n > 1.$

$h_n \leq \sqrt{2} \frac{1}{\sqrt{n}} \qquad \forall n > 1.$

$0 < n^{1/n} - 1 = h_n \leq \sqrt{2} \frac{1}{\sqrt{n}} \qquad \forall n > 1.$

$|n^{1/n} - 1| \leq \sqrt{2} \cdot \frac{1}{\sqrt{n}} \qquad \forall n > 1.$

By above theorem

$\lim_{n \to \infty} n^{1/n} = 1$

...eorem # Every Convergent Sequence is ...nded.

...roof # Let $\{a_n\}$ be an arbitrary ...vergent sequence of real number and

$\lim_{n \to \infty} a_n = A$

For $\epsilon = 1$ $\exists$ a natural no m ... that

$|a_n - A| < 1 \qquad \forall n \geq m. \longrightarrow (1)$

$\ldots| = |a_n - A + A|$

$\leq |a_n - A| + |A|$

$< 1 + |A| \qquad \forall n \geq m.$

$|a_n| < 1+|A| \quad \forall n \geq m \rightarrow (2)$

Consider the set

$\{|a_1|, |a_2|, |a_3|, \dots, |a_{m-1}|\} = \{|a_n| : n \leq m-1\}$

Which is finite and has a maximum.

Let $M_1 = \max\{|a_n| : n \leq m-1\}$

Then $|a_n| \leq M_1 \quad \forall n \leq m-1 \rightarrow (3)$

Let $M = \max\{1+|A|, M_1\}$

Then by (2)

$|a_n| < 1+|A| \leq M \quad \forall n \geq m \rightarrow (4)$

By (3)

$|a_n| \leq M_1 \leq M \quad \forall n \leq m-1 \rightarrow (5)$

From (4) & (5) we have.

$|a_n| \leq M \quad \forall n \in N$

$\Rightarrow \{a_n\}$ is bounded.

<u>OR</u>

From (1)

$A-1 < a_n < A+1 \quad \forall n \geq m$

Let $k = \min\{a_1, a_2, a_3 \dots a_{m-1}, A-1\}$

and $K = \max\{a_1, a_2, a_3 \dots a_{m-1}, A+1\}$

Then. $k \leq a_n \quad n \leq m-1 \rightarrow (1)$

and. $k \leq A-1 < a_n \quad \forall n \geq m \rightarrow (2)$

$\Rightarrow k \leq a_n \quad \forall n \rightarrow (3)$ by (1) & (2)

Also

$a_n \leq K \quad \forall n \leq m-1$

and $a_n < A+1 \leq K \quad \forall n \geq m.$

$\Rightarrow \quad a_n \leq K \quad \forall n \quad \forall n \geq m \longrightarrow (5)$

By (4) & (5)

$k \leq a_n \leq K \quad \forall n.$

$\Rightarrow \{a_n\}$ is bounded.

<u>Discussion</u> #(1) The intuition for this theorem is quite simple. First by definition of limit all the terms with large index must be close to A). Since the no of terms with small index is finite, and every finite set is bounded, therefore we are able to construct a bound for all terms of the sequence.

(2) # The converse of the above theorem is not true i.e a bounded sequence is not necessarily convergent e.g the sequence $\{(-1)^n\}$ is bounded and divergent.

(3) Convergence $\longrightarrow$ Boundedness

Contra-positive of it is

A sequence that is not bounded can never converge.

This is a useful tool for showing certain sequences do not converge. The sequence $\{n\}$ diverges because the set of +ve integers is not bounded.

Theorem # If a sequence is unbounded, then it must diverge or it can not converge.

Proof # Suppose that the sequence $\{a_n\}$ is unbounded and let on the contrary it converges and its limit is A.

Then for $\epsilon = 1$ $\exists$ a natural no m such that

$$|a_n - A| < 1 \quad \forall n \geq m.$$

$$|a_n| = |a_n - A + A| \leq |a_n - A| + |A| \leq 1 + |A| \quad \forall n \geq m.$$

If $n < m$, then

$$|a_n| \leq Max\{|a_1|, |a_2|, \dots, |a_{m-1}|\}$$

Let $M = Max\{|a_1|, |a_2|, \dots, |a_{m-1}|, 1 + |A|\}$

Then $|a_n| \leq M \quad \forall n \in N$

$\Rightarrow \{a_n\}$ is bounded which is a contradiction.

Hence $\{a_n\}$ is not convergent.

Theorem # If $\{a_n\}$ is a convergent sequence of real numbers such that $a_n \geq 0$ $\forall n$ and $\lim_{n \to \infty} a_n = A$, then $A \geq 0$

Proof # Let on contrary $A < 0$

Then $-A > 0$

$\because \lim_{n \to \infty} a_n = A$

$\therefore$ For $\epsilon = -A > 0$ $\exists$ a natural no n_1 such that

$$|a_n - A| < -A = \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow \quad A - \epsilon < a_n < A + \epsilon \quad \forall n \geq n_1$$

In particular, we have

$$a_{n_1} < A + \epsilon = A + (-A) = 0$$

$$\Rightarrow a_{n_1} < 0$$

But this contradicts the the hypothesis that $a_n \geq 0$ $\forall n$. Thus $A \geq 0$

OR

For $\epsilon = -\frac{A}{2} > 0$ $\exists$ a natural no n_1 such that

$$|a_n - A| < -\frac{A}{2} = \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow \quad a_n - A < -A/2 \quad \forall n \geq n_1$$

$$\Rightarrow \quad a_n < \frac{A}{2} < 0 \quad \forall n \geq n_1$$

But by hypothesis $a_n \geq 0$ $\forall n$.

Hence a contradiction. Thus $A \geq 0$

Note (1) This theorem states that a ~~the~~ sequence of non-negative terms if converges, then converges

to a non-negative limit.

(2) If the sequence ultimately becomes non-negative, then if exists, it will have non-negative limit.

Theorem # If a sequence $\{a_n\}$ converges ~~to~~ to A, then the sequence $\{|a_n|\}$ converges to $|A|$

Proof # $\because$ $\{a_n\}$ converges to A

$\therefore$ For a given $\epsilon > 0$ $\exists$ a natural no n_1 such that

$$|a_n - A| < \epsilon \qquad \forall n \geq n_1$$

Now

$$\big||a_n| - |A|\big| \leq |a_n - A| < \epsilon \qquad \forall n \geq n_1$$

$\Rightarrow$ $\{|a_n|\}$ converges to $|A|$

Converse of above is not true $|a_n| = 1$ cgt but $\{(-1)^n\}$ is dgt

---

Theorem Let $\{x_n\}$ be a sequence of real nos. that converges to x and suppose that $x_n \geq 0$. Then the sequence $\{\sqrt{x_n}\}$ of the square roots converges and $\lim\limits_{n\to\infty}(\sqrt{x_n}) = \sqrt{x}$

Proof # $\because$ $x_n \geq 0$ $\therefore$ $\lim\limits_{n\to\infty} x_n \geq 0$

So The theorem makes the sense.

Case (i) if $x = 0$, then $x_n \longrightarrow 0$ and for given $\epsilon > 0$ $\exists$ a natural no n_1 such that

$$|x_n - 0| < \epsilon^2 \qquad \forall n \geq n_1$$

$0 \leq x_n$

$\Rightarrow 0 \leq \sqrt{x_n}$

$\because \epsilon$ is arbit

$\therefore \sqrt{x_n} -$

<u>Case ii</u> if $x -$

$\sqrt{x_n} - \sqrt{x} =$

$=$

$\because \sqrt{x_n} + \sqrt{x} \geq$

$\therefore \frac{1}{\sqrt{x_n}+\sqrt{x}} \leq$

and

$\sqrt{x_n} - \sqrt{x} = \frac{x_n -}{\sqrt{x_n}}$

$\Rightarrow |\sqrt{x_n} - \sqrt{x}| \leq$

$\because x_n \longrightarrow x$

$\therefore$ for $\epsilon\sqrt{x} > 0$

Such that

$|x_n - x|$

using in 1

$|\sqrt{x_n} - \sqrt{x}| <$

Theorem # Let $\{a_n\}$ be a sequence. $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} |a_n| = 0$ i.e $\{a_n\}$ is a null sequence iff $\{|a_n|\}$ is a null sequence.

Proof # Suppose that $\{a_n\}$ is a null sequence, then $\lim_{n\to\infty} a_n = 0$

$\therefore$ Given $\epsilon > 0$ $\exists$ a +ve integer n_1 such that

$$|a_n - 0| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow |a_n| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow ||a_n| - 0| = ||a_n|| = |a_n| < \epsilon \quad \forall n \geq n_1$$

$\Rightarrow \{|a_n|\}$ is a null sequence.

Now Suppose that $\{|a_n|\}$ is a null sequence, then $\lim_{n\to\infty} |a_n| = 0$

$\therefore$ Given $\epsilon > 0$ $\exists$ a natural no m such That

$$||a_n| - 0| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow ||a_n|| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow |a_n| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow |a_n - 0| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

Theorem# If $\{a_n\}$ is a null sequence and $\{b_n\}$ is a bounded sequence, then $\{a_n b_n\}$ is a null sequence.

Proof# $\because$ $\{b_n\}$ is a bounded sequence

$\therefore$ $\exists$ a real number M such that

$$|b_n| \leq M \quad \forall n.$$

Also $\{a_n\}$ is a null sequence

$\Rightarrow$ Given $\epsilon > 0$, $\exists$ a +ve integer m such that $\forall n \geq m$

$$|a_n| < \epsilon/M$$

$$|a_n b_n - 0| = |a_n|\,|b_n|$$
$$< \epsilon/M \cdot M = \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n\to\infty} a_n b_n = 0$$

Theorem# If a sequence $\{a_n\}$ oscillates finitely and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} (a_n b_n) = 0$

Proof# $\because$ $\{a_n\}$ oscillates finitely

$\therefore$ $\{a_n\}$ is bounded.

By above theorem $\lim_{n\to\infty} a_n b_n = 0$

Theorem# If $\{a_n\}$ is a null sequence and c is a constant, then $\{c a_n\}$ is a null sequence

Proof # $\because$ $\{a_n\}$ is a null sequence

$\Rightarrow$ Given $\epsilon > 0$, $\exists$ a +ve integer m.

Such that
$$|a_n| < \frac{\epsilon}{|c|+1} \quad \forall n \geq m$$

Now
$$|ca_n| = |c|\,|a_n| < \left(\frac{|c|}{|c|+1}\right)\epsilon < \epsilon \quad \forall n \geq m$$

$\Rightarrow \lim\limits_{n \to \infty} ca_n = 0$

$\Rightarrow \{ca_n\}$ is a null sequence.

Subsequence

If a new sequence is constructed from the terms of old sequence by picking out terms in any way (but preserving the original order) but in the same order as in original, then new sequence is called a subsequence of the old sequence.

OR

Let $f: N \to N$ be a strictly increasing function with $f(k)$ denoted by n_k. If $\{a_n\}$ is any sequence, then $\{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\}$ is called a subsequence of $\{a_n\}$.

Note $\quad a: N \to R \qquad f: N \to N$

$$(a \circ f)(k) = a_{f(k)} = a_{n_k}.$$

Explanation

Consider the sequence $\{a_n\}$ defined by $a_n = \frac{1}{n}$ i.e

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \text{-------} \longrightarrow (1)$$

If we construct a subsequence by crossing out every other term, we get a subsequence.

$$1, \tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{5}, \text{-------}$$

Sub-sequence index $\xrightarrow{f}$ Super sequence index N $\overset{a}{\dashrightarrow}$ term.

N

$k \xrightarrow{f} f(k) = n_k$.

$1 \xrightarrow{f} f(1) = n_1 = 1 \longrightarrow a_1 = 1$

$2 \xrightarrow{f} f(2) = n_2 = 3 \longrightarrow a_3 = \tfrac{1}{3}$

$3 \longrightarrow f(3) = n_3 = 5 \longrightarrow a_5 = \tfrac{1}{5}$

$k \xrightarrow{f} f(k) = n_k = 2k-1 \longrightarrow a_{2k-1} = \frac{1}{2k-1}$

Thus subsequence $\{a_{n_k}\} = \{a_{2k-1}\} = \left\{\frac{1}{2k-1}\right\}_{k=1}^{\infty}$

or subsequence is $\left\{\frac{1}{2n-1}\right\}_{n=1}^{\infty}$

Note (1) we note that n_k is no of index which corresponds to kth term of the sub sequence.

(2) $n_k \geq k$

If we cross-out every odd numbered term we get the subsequence

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$ - - - - - - - -

Sub-Sequence index		(n_k) Sequence index	$\xrightarrow{a}$	term Sub-sequence a_{n_k}
1	$\longrightarrow$	$n_1 = 2$	$\xrightarrow{a}$	$a_2 = 1/2$
2	$\longrightarrow$	$n_2 = 4$	$\xrightarrow{a}$	$a_4 = \frac{1}{4}$
3	$\longrightarrow$	$n_3 = 6$	$\xrightarrow{a}$	$a_6 = \frac{1}{6}$
- - -		- - -		- - -
- - -		- - -		- - -
k	$\longrightarrow$	$n_k = 2k$	$\longrightarrow$	$a_{2k} = \frac{1}{2k}$

Thus the subsequence is $\left\{\frac{1}{2k}\right\}_{k=1}^{\infty} = \left\{\frac{1}{2n}\right\}_{n=1}^{\infty}$

Examples

The subsequences of the sequence of eve integers $\{n\}$ are.

(a) The sub-sequence of even integers $= \{2n\}$
$= \{2, 4, 6, \ldots\}$

(b) The sub-sequence of odd integers $= \{2n-1\}_{n=1}^{\infty}$
$= \{1, 3, 5, \ldots$

(c) The subsequence of primes $2, 3, 5, 7, 11 \ldots$

Remarks # (1) # The terms of a subsequence occur in the same order in which they occur in the original sequence.

(2) # Every sequence is a subsequence of itself

(3) # The interval in the various terms of a subsequence need not be regular.

(4) # Given a term a_m of a sequence $\{a_n\}$, there is a term of the subsequence following it.

Theorem # If a sequence $\{a_n\}$ converges to A, then every subsequence of $\{a_n\}$ converges to A.

Proof # Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$

$\because$ $\{a_n\}$ converges to A

$\therefore$ Given $\epsilon > 0$, $\exists$ a natural no N_1 such that

$$|a_n - A| < \epsilon \qquad \forall n \geq N_1 \rightarrow ①$$

Since n_k is strictly increasing sequence and $n \geq N_1$, therefore $n_n \geq n \geq N_1$. Thus if $k \geq N_1$, then we have $n_k \geq k \geq N_1$ and from ①

$$|a_{n_k} - A| < \epsilon \qquad \forall n_k \geq N_1$$

$\Rightarrow \{a_{n_k}\}$ converges to A.

Note # $\boxed{1}$ # The converse of the above theorem is not true i.e if a subsequence or even an infinitely many subsequences of a given sequence converge, the original sequence may not converge. e.g let $a_n = (-1)^n$. Then $\{a_n\}$ does not converge. However the two subsequences $\{a_{2n-1}\}$ and $\{a_{2n}\}$ converge to -1 and 1 respectively.

$\boxed{2}$ # If all subsequences of a sequence $\{a_n\}$ converge to the same limit, only then $\{a_n\}$ converges to that limit.

$\boxed{3}$ # To prove that a sequence is not convergent it is sufficient to show that two of its subsequences converge to different limit.

In fact $\lim\limits_{n\to\infty} a_n = A$ iff every subsequence converges to the same limit

Example

The sequences $\{\frac{1}{2^n}\}$ & $\{\frac{1}{n!}\}$ are subsequences of the convergent sequence $\{\frac{1}{n}\}$ and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Thus $\lim\limits_{n\to\infty} \frac{1}{2^n} = \lim\limits_{n\to\infty} \frac{1}{n!} = 0$

Theorem # *(extra) A sequence $\{a_n\}$ converges to a limit A iff the subsequences of even numbered terms and odd numbered terms i.e $\{a_{2n}\}$ & $\{a_{2n-1}\}$ both converge to A.

Proof # Let $\{a_n\}$ converges to A. Then for

given $\epsilon > 0$, $\exists$ a natural no N_1 such that

$$|a_n - A| < \epsilon \quad \forall n \geq N_1$$

$$\Rightarrow |a_{2n} - A| < \epsilon \quad \forall (2n) \geq N_1.$$

and $$|a_{2n-1} - A| < \epsilon \quad \forall (2n-1) \geq N_1.$$

$\Rightarrow$ Subsequences $\{a_{2n}\}$ & $\{a_{2n-1}\}$ converge to A

<u>Converse</u> Let $\{a_{2n}\}$ & $\{a_{2n-1}\}$ both converge to A. Then for given $\epsilon > 0$ $\exists$ natural nos N_1 & N_2 such that

$$|a_{2n} - A| < \epsilon \quad \forall\, 2n \geq N_1$$

and $$|a_{2n-1} - A| < \epsilon \quad \forall (2n-1) \geq N_2.$$

Let $N_3 = \max(N_1, N_2)$, then

$$|a_{2n} - A| < \epsilon \quad \forall (2n) \geq N_3$$

and $$|a_{2n-1} - A| < \epsilon \quad \forall (2n-1) \geq N_3$$

$$\Rightarrow |a_n - A| < \epsilon \quad \forall n \geq N_3$$

$\Rightarrow$ $\{a_n\}$ converges to A.

<u>Examples</u>

The sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \frac{1}{2^4}, \frac{1}{3^4}, \cdots$$

converges to zero since even numbered and odd numbered terms both converge to 0.

Sequence $1, 1/2, 1, 1/3, 1, 1/4, \cdots$ diverges because even and odd numbered terms converge to 0 & 1

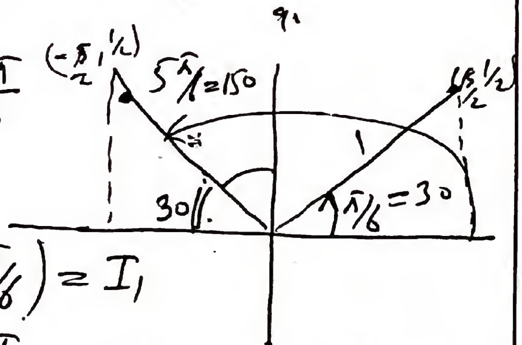
Q# Let a

Q# prove that sequence $\{\sin n\}$ diverges

Sol We use elementry properties of sine function.

We note that

$\sin\frac{\pi}{6} = \frac{1}{2} = \sin\frac{5\pi}{6}$



and

$\sin x >$ in interval $(\pi/6, 5\pi/6) = I_1$

$\because$ The length of the interval $I_1 = 5\pi/6 - \pi/6 = 2\pi/3 > 2$

$\therefore$ There are at least two natural numbers lying inside I_1. let n_1 be the 1st such number.

Similarly for each $k \in N$

$\sin x > \frac{1}{2} \quad \forall x \in \left(\frac{\pi}{6} + 2\pi(k-1), \frac{5\pi}{6} + 2\pi(k-1)\right)$.

Let $I_k = \left(\frac{\pi}{6} + 2\pi(k-1), \frac{5\pi}{6} + 2\pi(k-1)\right)$

$\because$ The length of I_k is greater than 2

$\therefore$ There are at least two natural numbers lying inside I_k.

we let n_k be the 1st one. The subsequence $\{\sin n_k\}$ of $\{\sin n\}$ obtained in this has property that

$\sin n_k \in [1/2 \ 1] \quad \forall n_k$

Similarly if $k \in N$ and J_k is the interval

$J_k = \left(\frac{7\pi}{6} + 2\pi(k-1), \frac{11\pi}{6} + 2\pi(k-1)\right)$, then

$\sin x < -1/2 \quad \forall x \in J_k$.

Theorem # (a) If a sequence $\{a_n\}$ diverges to $+\infty$, then every subsequence of $\{a_n\}$ also diverges to $+\infty$.

(b) If a sequence $\{a_n\}$ diverges to $-\infty$, then every subsequence of $\{a_n\}$ also diverges to $-\infty$.

Proof # Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$.

$\because$ $\{a_n\}$ diverges to $+\infty$

$\therefore$ For every +ve real no K, however large, $\exists$ a natural no m such that

$$a_n > K \quad \forall n \geq m.$$

$\because$ $\{n_k\}$ is strictly increasing sequence

$\therefore$ For $k_0 \geq m$, we have $n_{k_0} \geq k_0 \geq m$.

and hence for $n_k \geq n_{k_0} \geq m$

$$a_{n_k} > K \quad \forall n_k \geq m.$$

$\Rightarrow$ $\{a_{n_k}\}$ diverges to $+\infty$

(b) Try yourself

Note $\boxed{1}$ # The converse of above theorem is not true i.e if a subsequence of a given sequence diverges to $+\infty(-\infty)$, then the sequence need not diverge to $+\infty$ (or $-\infty$) e.g

let $a_n = (-1)^n n = \begin{cases} -n & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$

Then sub-sequence $\{a_{2n-1}\}$ diverges to $-\infty$ & the subsequence $\{a_{2n}\}$ diverges to $+\infty$ but the sequence does not diverge to $+\infty$ or $-\infty$ but oscillates.

$\boxed{2}$ # If all subsequences diverge to $+\infty(-\infty)$ only then the sequence diverges to $+\infty(-\infty)$.

Q# Let a be a ...

64

44

$\because$ length of J_k is greater than 2.

$\therefore$ There are atleast two natural numbers lying inside J_k. Let m_k be the 1st natural number lying in J_k.

The subsequence $\{\sin m_k\}$ of $\{\sin n\}$ is such that

$$\sin m_k \in [-1, -\tfrac{1}{2}] \quad \forall\, m_k.$$

Now $[-1, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, 1]$

let c be any real number. Then at least of the subsequences $\{\sin n_k\}$ & $\{\sin m_k\}$ lie entirely outside of $\frac{1}{2}$-nbhd of c i.e $(c-\tfrac{1}{2}, c+\tfrac{1}{2})$. Therefore c can not be a limit of that subsequence. Since $c \in R$ is an arbitrary, therefore that subsequence is divergent and the sequence is also divergent.

Q# A sequence defined by

$$a_n = \left(1 - \frac{1}{n}\right)\sin\frac{n\pi}{2}$$

diverges

Sol We have.

$$a_{2k} = \left(1 - \frac{1}{2k}\right)\sin k\pi = \left(1 - \frac{1}{2k}\right)\cdot 0 = 0$$

$k = 1, 2, \ldots$

$$a_{4k+1} = \left(1 - \frac{1}{4k+1}\right)\sin(4k+1)\frac{\pi}{2}$$

$$= \left(1 - \frac{1}{4k+1}\right)(1) \qquad k = 1, 2, \ldots$$

We note that subsequences $\{a_{2k}\}$ and $\{a_{4k+1}\}$ converges to 0 & 1

Convergence of Constant Sequence.

Theorem# A constant sequence is cgt.

OR

If $a_n = c \quad \forall n \in N$, then

$$\lim_{n\to\infty} a_n = c.$$

Proof# Let $\epsilon > 0$, then.

$$|a_n - c| = |c - c| = 0 < \epsilon \quad \forall n \geq 1 = n_1$$

$$\Rightarrow a_n \longrightarrow c$$

ALGEBRA OF LIMITS#

If $\{a_n\}$ & $\{b_n\}$ are sequences of real nos, we define their sum to be.

$\{a_n + b_n\}$ addition is performed term by term.

Product by $\{a_n b_n\}$

Difference by $\{a_n - b_n\}$

If $b_n \neq \quad \forall n \in N$, then division is defined by

$$\frac{\{a_n\}}{\{b_n\}} = \left\{\frac{a_n}{b_n}\right\}$$

Multiple of sequence $\{a_n\}$ by $c \in R$ is defined by $\{c a_n\}$

64

Q# Let a be a Constant. Then

46

Theorem# If $\{a_n\}$ & $\{b_n\}$ be sequence of real nos. that converge to A & B respectively and $c \in R$. Then.

(a) $\lim_{n\to\infty} (a_n+b_n) = A+B$ (a₁) $\lim_{n\to\infty} ca_n = cA$

(b) $\lim_{n\to\infty} (a_n-b_n) = A-B$

(c) $\lim (a_nb_n) = AB$

(d) If $a_n \neq 0 \ \forall n$ and $A \neq 0$, then

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}$$

(e) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ provided $b_n \neq 0$ & $B \neq 0$

Proof# (a₁) $\because \lim_{n\to\infty} a_n = A$

$\therefore$ Given $\epsilon > 0$ $\exists$ even integer m such that

$$|a_n - A| < \frac{\epsilon}{|c|+1} \qquad \forall n \geq m$$

$$|ca_n - cA| = |c|\,|a_n - A|$$

$$< |c| \frac{\epsilon}{|c|+1} < \epsilon \qquad \forall n \geq m$$

$$\Rightarrow \lim_{n\to\infty} ca_n = cA$$

OR

If $c = 0$, then result is obvious because

$|ca_n - cA| = 0 < \epsilon \quad \forall n \quad \forall \epsilon > 0$

Let $c \neq 0$.

$a_n \longrightarrow A$

$\therefore$ for given $\epsilon > 0$ $\exists$ a natural no m

Such that

$$|a_n - A| < \frac{\epsilon}{|c|}$$

$$|ca_n - cA| = |c|\,|a_n - A|$$

$$< |c| \frac{\epsilon}{|c|} = \epsilon \quad \forall n \geq m$$

$\Rightarrow$ $|ca_n - cA| < \epsilon \quad \forall n \geq m$

$\Rightarrow$ $\lim_{n\to\infty} ca_n = cA$

(a) Given $\epsilon > 0$

$\because$ $\lim_{n\to\infty} a_n = A \qquad \lim_{n\to\infty} b_n = B$

$\therefore$ $\exists$ natural nos m_1 & m_2 such that

$|a_n - A| < \epsilon/2 \quad \forall n \geq m_1 \longrightarrow$ (1)

$|b_n - B| < \epsilon/2 \quad \forall n \geq m_2 \longrightarrow$ (2)

Let $m = \max(m_1, m_2)$. Then

$|a_n - A| < \epsilon/2 \longrightarrow$ (3)

$|b_n - B| < \epsilon/2 \longrightarrow$ (4)

$$|(a_n + b_n) - (A+B)| = |(a_n - A) + (b_n - B)|$$

$$\leq |a_n - A| + |b_n - B|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \forall n \geq m$$

$\Rightarrow$ $\lim_{n\to\infty} (a_n + b_n) = A+B$

Let a be a Constant. Then

(b) $|(a_n+b_n)-(A-B)| = |(a_n-A)-(b_n-B)|$

$\leq |a_n-A|+|b_n-B|$

$< \epsilon/2 + \epsilon/2 = \epsilon \quad \forall n \geq m$

$\Rightarrow |(a_n+b_n)-(A-B)| < \epsilon \quad \forall n \geq m$

$\Rightarrow \lim_{n\to\infty}(a_n+b_n) = A-B$

Note The converse of (a) & (b) not necessarily true i.e existence $\lim_{n\to\infty}(a_n \pm b_n)$ does not necessarily imply that $\lim a_n$ & $\lim b_n$ also exist. e.g let $a_n = n$. $b_n = -n$ both divergent but $a_n+b_n = 0 \ \forall n$ & $\lim_{n\to\infty}(a_n+b_n)=0$

let $a_n = n$ $\quad b_n = n$

$a_n - b_n = 0$ & $\lim_{n\to\infty}(a_n-b_n) = 0$ but

$\{a_n\}$ & $\{b_n\}$ are divergent

(c) Let $\epsilon > 0$ be given.

$|a_n \cdot b_n - AB| = |a_n \cdot b_n - b_n \cdot A + b_n A - AB$

$\leq |a_n \cdot b_n - b_n \cdot A| + |b_n A - AB|$

$= |b_n|\,|a_n - A| + |A|\,|b_n - B| \longrightarrow (1)$

$\because$ The sequence $\{b_n\}$ being cgt is bounded.

$\therefore \exists$ a no M such that

$|b_n| \leq M$

$\because \lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$

$\therefore \exists$ nos $m_1, m_2 \in N$ such that

$|a_n - A| < \frac{\epsilon}{2M} \quad \forall n \geq m_1 \longrightarrow$

and $|b_n - B| < \frac{\epsilon}{2(|A|+1)} \quad \forall n \geq m_2$

[$\frac{\epsilon}{2(|A|+1)}$ used. rather than $\frac{\epsilon}{2|A|}$ to avoid. from $A=0$

Let $m = \max(m_1, m_2)$. Then.

$|a_n - A| < \frac{\epsilon}{2M} \longrightarrow (2)$

$|b_n - B| < \frac{\epsilon}{2(|A|+1)} \longrightarrow (3)$

From (1), (2) & (3)

$|a_n \cdot b_n - AB| \leq |b_n||a_n - A| + |A||b_n - B|$

$< M \cdot \frac{\epsilon}{2M} + |A| \cdot \frac{\epsilon}{2(|A|+1)} \quad \forall n \geq m$

$= \frac{\epsilon}{2} + \left(\frac{|A|}{|A|+1}\right) \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon \quad \forall n \geq m$

$\Rightarrow |a_n \cdot b_n - AB| < \epsilon \quad \forall n \geq m$

$\Rightarrow \lim_{n\to\infty} a_n b_n = AB$.

Q# Let a 64 be a Constant. Then

50

Corollary# Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} a_n = l$ and k be any +ve integer. Then $\lim_{n\to\infty} a_n^k = l^k$.

Proof# By induction on k

For $k = 2$

$$\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} a_n \cdot a_n = l \cdot l = l^2$$

$\Rightarrow$ It is true for $k = 2$

Let it be true for $k = p$ ie

$$\lim_{n\to\infty} a_n^p = l^p \qquad p > 2$$

$\Rightarrow \cancel{l} \cdot \lim a_n^p = l \cdot l^p$

$\Rightarrow \lim_{n\to\infty} a_n \cdot \lim a_n^p = l^{p+1}$

$\Rightarrow \lim_{n\to\infty} (a_n \cdot a_n^p) = l^{p+1}$

$\Rightarrow \lim_{n\to\infty} a_n^{p+1} = l^{p+1}$

$\Rightarrow$ It is true for $k = p+1$

Thus it is true for all +ve integral values of k and

$$\lim_{n\to\infty} a_n^k = l^k$$

Note the converse of the above theorem is not necessarily true i.e existence $\lim_{n\to\infty}(a_n b_n)$ does not necessarily imply that the two limits $\lim_{n\to\infty} a_n$ & $\lim_{n\to\infty} b_n$ also exist. e.g

let $a_n = b_n = (-1)^n$, then both limits $\lim_{n\to\infty} a_n$ & $\lim_{n\to\infty} b_n$ do not exist. But

$$a_n b_n = (-1)^n(-1)^n = (-1)^{2n} = 1 \quad \forall n.$$

So that $\lim_{n\to\infty}(a_n b_n) = 1$ exists.

(d) Fix $\epsilon > 0$.

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| = \left|\frac{A - a_n}{a_n A}\right| = \frac{|a_n - A|}{|a_n|\,|A|} \rightarrow (1)$$

$\because \lim_{n\to\infty} a_n = A \quad \therefore \lim_{n\to\infty} |a_n| = |A|$

Taking $\epsilon = \frac{|A|}{2}$

$\exists$ a +ve integer m_1 s.that

$$\big||a_n| - |A|\big| < \frac{|A|}{2} \quad \forall n \geq m_1$$

$$\Rightarrow |A| - \frac{|A|}{2} < |a_n| < |A| + \frac{|A|}{2} \quad \forall n \geq m_1$$

$$\frac{|A|}{2} < |a_n| \quad \forall n \geq m_1$$

49

Q# Let a be a constant. Then

64

54

53

$$|a_n - A| < \frac{|A|}{2} \quad \forall n \geq m_1$$

Now

$$||a_n| - |A|| \leq |a_n - A| < \frac{|A|}{2} \quad \forall n \geq m_1$$

$$\Rightarrow -\frac{|A|}{2} \leq -|a_n - A| \leq |a_n| - |A| \quad \forall n \geq m_1$$

$$\leq |a_n| - |A|$$

$$\Rightarrow -\frac{|A|}{2} \leq |a_n| - |A|$$

$$\Rightarrow \frac{|A|}{2} \leq |a_n|$$

$$\Rightarrow \frac{1}{|a_n|} \leq \frac{2}{|A|} \quad \forall n \geq m_1$$

OR

$$|a_n - A| < \frac{|A|}{2} \quad \forall n \geq m_1$$

$$\Rightarrow |a_n - A| + |a_n| < \frac{|A|}{2} + |a_n| \quad \forall n \geq m_1 \longrightarrow (a)$$

$$|A| = |A - a_n + a_n| \leq |A - a_n| + |a_n|$$

$$\Rightarrow |A| \leq |A - a_n| + |a_n| < \frac{|A|}{2} + |a_n| \quad \forall n \geq m_1$$

$$\Rightarrow \frac{|A|}{2} < |a_n|$$

$$\Rightarrow \frac{1}{|a_n|} < \frac{2}{|A|} \quad \forall n \geq m_1$$

$$\frac{2}{|A|} > \frac{1}{|a_n|} \qquad \forall\ n \geq m_1$$

$$\frac{1}{|a_n|} < \frac{2}{|A|} \qquad \forall\ n \geq m_1$$

Also $\lim_{n\to\infty} a_n = A$, therefore $\exists$ natural no m_2 s. that

$$|a_n - A| < \frac{|A|^2\epsilon}{2} \qquad \forall\ n \geq m_2$$

Let $m = \max(m_1, m_2)$. Then.

$$\frac{1}{|a_n|} < \frac{2}{|A|} \qquad \forall\ n \geq m \longrightarrow (2)$$

and $$|a_n - A| < \frac{|A|^2\epsilon}{2} \qquad \forall\ n \geq m \longrightarrow (3)$$

From (1) (2) & (3)

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| = \frac{|a_n - A|}{|a_n|\,|A|}$$

$$< \frac{2}{|A|} \cdot \frac{1}{|A|} \cdot \frac{|A|^2\epsilon}{2} = \epsilon \qquad \forall\ n \geq m.$$

$$\Rightarrow \left|\frac{1}{a_n} - \frac{1}{A}\right| < \epsilon \qquad \forall\ n \geq m.$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}$$

<u>OR</u>

$\because \lim_{n\to\infty} a_n = A$

$\therefore$ For $\epsilon = \frac{|A|}{2} > 0$ $\exists\ m_1 \in N$ s. that

Q#

Remarks

terms ... zero

essential ... is that

... use then

non zero ... n-zero

eventually ... verses {

limit of ... > 0 be g

2) # Le

$\frac{a_n}{b_n} - \frac{A}{B}\Big| = \Big|$ ... $Abn \Big| = \Big| \frac{a}{}$

$= \frac{|B(a_n -}{|b_n}$ ... $A(B - b_n)| \leq$

$= \frac{|B|\,|a}{|b_n}$ ... $+ |A|\,|b_n -$

$= \frac{|a_n -}{|b_n|}$ ... $+ \frac{|A|\,|b_n}{|b_n|\,|B|}$

$\lim_{n\to\infty} b$ ... B

$\exists$ a n ... no m_1

$|b_n - B|$ ... $\frac{|B|}{2}$

$\frac{|B|}{2} > |$ ... $\geq |B|$

$|b_n| >$ ... A

$|b_n| <$ ... $\forall n$

OR

$$||b_n| - |B|| \leq |b_n - B| < \frac{|B|}{2} \quad \forall n \geq m_1$$

$$\Rightarrow |B| - \frac{|B|}{2} < |b_n| < |B| + \frac{|B|}{2} \quad " \quad "$$

$$\Rightarrow |b_n| > \frac{|B|}{2} \quad \forall n \geq m_1$$

$$\Rightarrow \frac{1}{|b_n|} < \frac{2}{|B|} \quad \forall n \geq m_1 \longrightarrow ②$$

Also $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$

$\Rightarrow \exists$ natural nos m_2, m_3 such that

$$|a_n - A| < \frac{|B|\epsilon}{4} \quad \forall n \geq m_2 \longrightarrow ③$$

$$|b_n - B| < \frac{|B|^2\epsilon}{4(|A|+1)} \quad \forall n \geq m_3 \longrightarrow ④$$

Let $m = \max(m_1, m_2, m_3)$. Then.

$$\frac{1}{|b_n|} < \frac{2}{|B|} \quad \forall n \geq m \longrightarrow ⑤$$

$$|a_n - A| < \frac{|B|\epsilon}{4} \quad " \quad " \longrightarrow ⑥$$

$$|b_n - B| < \frac{|B|^2\epsilon}{4(|A|+1)} \quad " \quad " \longrightarrow ⑦$$

From ①, ⑤, ⑥ & ⑦

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| \leq \frac{|a_n - A|}{|b_n|} + \frac{|A|\,|b_n - B|}{|b_n|\,|B|}$$

$$< \frac{2}{|B|} \cdot \frac{|B|\epsilon}{4} + \frac{2}{|B|} \cdot \frac{|A|}{|B|} \cdot \frac{|B|^2\epsilon}{4(|A|+1)} \quad \forall n \geq m$$

64

Q# Let a be a Constant.

55+1 = 56

$$= \frac{\epsilon}{2} + \frac{|A|}{|A|+1} \cdot \frac{\epsilon}{2} \quad \forall n \geq m.$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq m.$$

$$\Rightarrow \left| \frac{a_n}{b_n} - \frac{A}{B} \right| < \epsilon$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$$

Note# The converse of the above theorem is not necessarily true i.e the existence of $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$ does not necessarily imply that the two limits $\lim_{n\to\infty} a_n$ & $\lim_{n\to\infty} b_n$ also exist.

e.g $a_n = b_n = n$, then $\lim a_n$, $\lim b_n$ does not exist. But $\frac{a_n}{b_n} = \frac{1}{1} = 1$ & $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

Theorem# If $\lim_{n\to\infty} a_n = A$ & $A \neq 0$, then $\exists$ a +ve integer k and natural no m such that $|a_n| > k > 0$ i.e sequence is eventually bounded away from zero

Proof# $\because A \neq 0 \quad \therefore |A| > 0$

Also $\lim_{n\to\infty} a_n = A$

$\Rightarrow$ For $\epsilon = \frac{|A|}{2}$, $\exists$ natural no. m s.t

$$|a_n - A| < \frac{|A|}{2} \quad \forall n \geq m$$

Now $|A| = |A - a_n + a_n|$

$\leq |\cancel{A}_n| + |a_n - A|$

$< |a_n| + \frac{|A|}{2} \quad \forall n \geq m.$

$\Rightarrow \frac{|A|}{2} < |a_n| \quad \forall n \geq m.$

$\Rightarrow |a_n| > \frac{|A|}{2} > 0 \quad \forall n \geq m.$

$\Rightarrow |a_n| > \frac{|A|}{2} = k \quad \forall n \geq m$

Theorem # (Limit is order preserving on convergent sequences)

If $\lim_{n \to \infty} a_n = A$ & $\lim_{n \to \infty} b_n = B$, $a_n \leq b_n \ \forall n$

Then $A \leq B$ or $\lim a_n \leq \lim b_n$.

Proof # Let $c_n = b_n - a_n \quad \forall n.$

Then $c_n \geq 0 \quad \forall n \quad \because b_n \geq a_n \ \forall n$

$\Rightarrow \lim_{n \to \infty} c_n \geq 0$ ✓

$\Rightarrow \lim_{n \to \infty} (b_n - a_n) \geq 0$

$\Rightarrow \lim_{n \to \infty} b_n - \lim_{n \to \infty} a_n \geq 0$

$\Rightarrow B - A \geq 0$

$\Rightarrow A \leq B$ proved

64

Q# Let a be a Constant Th...

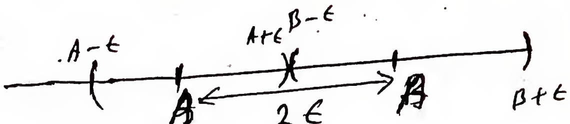
$57+1=58$

OR

On Contrary let $A > B$ and

$A - B = 3\epsilon$

So that nbhds



$]B-\epsilon, B+\epsilon[$ & $]A-\epsilon, A+\epsilon[$

of B and A are disjoint

$\because$ $a_n \longrightarrow A$ & $b_n \longrightarrow B$

$\therefore$ $\exists$ natural no m_1 & m_2 such that

$A-\epsilon < a_n < A+\epsilon \quad \forall n \geq m_1$

$B-\epsilon < b_n < B+\epsilon \quad \forall n \geq m_2$

Let $m = \max(m_1, m_2)$

Then $a_n \in\ ]A-\epsilon, A+\epsilon[\quad \forall n \geq m$

$b_n \in\ ]B-\epsilon, B+\epsilon[\quad \forall n \geq m$

$\Rightarrow b_n < a_n \quad \forall n \geq m$

Which Contradicts the fact That

$a_n \leq b_n \quad \forall n$

Hence our supposition is wrong and

$A \leq B$

Corollary If $a_n \leq k \ \forall n$ & $\lim_{n \to \infty} a_n = A$, then $A \leq k$.

Let $b_n = k - a_n$. Then $b_n \geq 0 \ \forall n$.

$\Rightarrow \lim b_n \geq 0 \Rightarrow \lim(k-a_n) \geq 0$

$\Rightarrow k - A \geq 0 \quad \Rightarrow k \geq A$.

$\underline{58}+1=\underline{59}$

Theorem# If $\{a_n\}$ be sequence such that

$$a \le a_n \le b \quad \forall n \in N$$

Then $$a \le \lim_{n\to\infty} a_n \le b$$

Proof# Consider sequence $\{b, b, b, \ldots\} = \{c_n\}$

Then $a_n \le c_n \quad \forall n$.

$$\Rightarrow \lim_{n\to\infty} a_n \le \lim_{n\to\infty} c_n$$

$$\Rightarrow \lim_{n\to\infty} a_n \le b \longrightarrow (1)$$

Consider sequence $\{d_n\} = \{a, a, a, \ldots\}$.

Then $d_n \le a_n \quad \forall n$.

$$\Rightarrow \lim_{n\to\infty} d_n \le \lim_{n\to\infty} a_n \longrightarrow (2)$$

$$a \le \lim_{n\to\infty} a_n$$

By (1) & (2)

$$a \le \lim_{n\to\infty} a_n \le b$$

Theorem# (Squeeze Theorem, Sandwhich Theorem) Trapped sequences (old squeeze play)

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be the sequences

Such that

$$a_n \le b_n \le c_n \quad \forall n.$$

(1) and $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$$

Then $$\lim_{n\to\infty} b_n = l.$$

(ii) For some +ve integer p,

$$a_n \leq b_n \leq c_n \quad \forall\ n \geq p$$

, then $\lim_{n\to\infty} b_n = l$

Proof # Let $\epsilon > 0$ be given.

$\because \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l.$

$\therefore \exists$ +ven integers m_1 & m_2 such that

$|a_n - l| < \epsilon \quad \forall n \geq m_1$

$|c_n - l| < \epsilon \quad \forall n \geq m_2$

Let $m = \max(m_1, m_2)$. Then.

$|a_n - l| < \epsilon \quad \forall n \geq m$

$|c_n - l| < \epsilon \quad \forall n \geq m$

$\Rightarrow l - \epsilon < a_n < l + \epsilon \quad \forall n \geq m.$

$l - \epsilon < c_n < l + \epsilon \quad \forall n \geq m$

Thus

$l - \epsilon < a_n \leq b_n \leq c_n < l + \epsilon \quad \forall n \geq m$

$\Rightarrow l - \epsilon < b_n < l + \epsilon \quad \forall n \geq m.$

$\Rightarrow |b_n - l| < \epsilon \quad \forall n \geq m.$

$\Rightarrow \lim_{n\to\infty} b_n = l.$

2

When $a_n \leq b_n \leq c_n \quad \forall n \geq p$

Then we take $m = \max(m_1, m_2, p)$

So that

$$|a_n - l| < \epsilon \quad \forall n \geq m$$

$$|c_n - l| < \epsilon \quad \forall n \geq m$$

& $$a_n \leq b_n \leq c_n \quad \forall n \geq m$$

$$\Rightarrow l - \epsilon < a_n \leq b_n \leq c_n < l + \epsilon \quad \forall n \geq m$$

$$\Rightarrow |b_n - l| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \to \infty} b_n = l.$$

Applications

Q1 # $\lim_{n \to \infty} \left(\frac{\sin n}{n}\right) = 0$

We can not apply $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = A/B$

because sequence $\{n\}$ is not convergent

However, we have

$$-1 \leq \sin n \leq 1 \quad \forall n$$

$$\Rightarrow -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \quad \forall n \in N$$

$\because \lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} \left(-\frac{1}{n}\right) = 0$

$\therefore$ By squeeze theorem.

Q# Let ... 64

Q# 63 Find $\lim\limits_{n\to\infty} a^{1/n}$, where a is a fixed +ve real number.

Sol # Case I # If $a \geq 1$, then

$$a^{1/n} \geq 1$$

Let $b_n = a^{1/n} - 1$

$$a^{1/n} = b_n + 1 \quad \text{where } b_n \geq 0$$

$\Rightarrow a = (1+b_n)^n \geq 1 + n b_n$ Bernoulli's inequality.

$\Rightarrow a - 1 \geq n b_n$

$\Rightarrow 0 \leq b_n \leq \frac{a-1}{n}$

$\because \lim\limits_{n\to\infty} \frac{a-1}{n} = (a-1) \lim\limits_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By squeeze play

$$\lim_{n\to\infty} b_n = 0$$

$\Rightarrow \lim\limits_{n\to\infty} a^{1/n} - 1 = 0$

$\Rightarrow \lim\limits_{n\to\infty} a^{1/n} = 1$

Case # 2 # $0 < a < 1$, then $1/a > 1$

and $\lim\limits_{n\to\infty} a^{1/n} = \lim \frac{1}{(1/a)^{1/n}} = \frac{1}{1} = 1$

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0$$

Q# (a) $\lim_{n\to\infty} \frac{\cos n}{n} = 0$

Sol $\because -1 \le \cos n \le 1$

$\therefore -\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$

$\Rightarrow$ By squeeze play

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0$$

or $\quad 0 \le \left|\frac{\cos n}{n}\right| \le \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} \left|\frac{\cos n}{n}\right| = 0$ by squeeze play

and hence $\lim_{n\to\infty} \frac{\cos n}{n} = 0$

(b) $(-1)^n \frac{1}{n} \to 0$

$\because \quad 0 \le \left|(-1)^n \frac{1}{n}\right| \le \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} (-1)^n \frac{1}{n} = 0$ by squeeze play

(c) $\frac{\cos^2 n}{3^n} \to 0$ because

$$0 \le \frac{\cos^2 n}{3^n} < \frac{1}{3^n}$$

$\Rightarrow \lim_{n\to\infty} \frac{\cos^2 n}{3^n} = 0$ by squeeze play

64

Q# Let a be a constant. Then prove that

(a) $a^n \longrightarrow 0$ if $0 < a < 1$.

(b) $a^n \longrightarrow +\infty$ if $a > 1$

Sol# $\because$ $0 < a < 1$

$\Rightarrow \frac{1}{a} > 1$

Let $\frac{1}{a} = 1 + p \qquad p > 0$

$\frac{1}{a^n} = (1+p)^n = 1 + np + (\text{+ve terms}) \geq np$

$\frac{1}{a^n} \geq np$

$\Rightarrow 0 < a^n < \frac{1}{np} = \frac{1}{p} \cdot \frac{1}{n}$

$\because \lim_{n \to \infty} \frac{1}{pn} = 0$

$\therefore$ By Squeeze play $\lim_{n \to \infty} a^n = 0$

(b) If $a > 1$, then

$a = 1 + b \qquad$ where $b > 0$

$a^n = (1+b)^n = 1 + nb + (\text{+ve terms}) > 1 + nb \to \infty$

$\Rightarrow a^n \longrightarrow +\infty$ as $n \longrightarrow \infty$

Ratio Test For Convergence

For certain types of sequences, the following result provide quick & easy ratio test for convergence.

Theorem # If $\{a_n\}$ be a sequence such that $a_n \neq 0$, be a sequence of +ve terms and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l \quad \text{where } l < 1$$

, then $\lim_{n\to\infty} a_n = 0$

Proof # $\because$ $a_n \geq 0$

$\therefore$ $\lim_{n\to\infty} a_n \geq$ & $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \geq 0$

$\Rightarrow l \geq 0$

Let k be an number such that

$$l < k < 1 \text{ and let } \epsilon = k - l > 0$$

Then $\exists$ a natural no m such that

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \frac{a_{n+1}}{a_n} < l + \epsilon = l + k - l = k \quad \forall n \geq m$$

$$\Rightarrow a_{n+1} < k a_n \longrightarrow (A) \quad \forall n \geq m$$

putting $n = m, m+1, m+2 \dots n-1$

we $a_{m+1} < k a_m \longrightarrow (1)$

$a_{m+2} < k a_{m+1} \longrightarrow (2)$

m

$n-m$

$m, m+1, \ldots, n-1$
$a_N = m+(N-1)$
$n-1 = m+N-1$
$n-m = N$

$a_{m+3} < k\, a_{m+2}$

$- \quad - \quad - \quad - \quad -$

$a_{n-1} < k\, a_{n-2}$

$a_n < k\, a_{n-1}$

Xing all above inequalities we get

$a_{m+1} a_{m+2} a_{m+3} \cdots a_{n-1} a_n$

$\leq k^{n-m} a_m a_{m+1} a_{m+2} \cdots a_{n-1}$

$\Rightarrow a_n < a_m k^{n-m}$

$\Rightarrow \dfrac{a_n}{a_m} < k^{n-m}$

$a_n < \dfrac{a_m k^n}{k^m} \longrightarrow$ B $\forall n \geq m$

But $0 < k < 1 \quad \therefore k^n \to 0$

Hence $\lim_{n \to \infty} a_n = 0$

<u>OR</u>

$\because k^n \to 0 \quad \therefore$ Given $\epsilon > 0\ \exists$

a natural no p such that

$|k^n - 0| < \dfrac{k^m \epsilon}{a_m} \quad \forall n \geq p \longrightarrow$ B

Let $m_1 = \max(m, p)$, then A & B both holds for $n \geq m_1$ and

67

$a_n < \dfrac{a_m r^n}{r^m} \longrightarrow$ (C) $\forall n \geq m_1$

$r^n < \dfrac{r^m \epsilon}{a_m} \longrightarrow$ (D) $\forall n \geq m_1$

$\Rightarrow a_n < \dfrac{a_m}{r^m} \times \dfrac{r^m \epsilon}{a_m} = \epsilon \quad \forall n \geq m_1$

$\Rightarrow \because a_n > 0$

$\Rightarrow |a_n - 0| < \epsilon \quad \forall n \geq m_1$

$\Rightarrow \lim\limits_{n \to \infty} a_n = 0$

OR

From (A)

$a_{n+1} < r a_n$

putting $n-1, n-2 \dots m-1$

$a_n < r a_{n-1}$

$a_{n-1} < r a_{n-2}$

$a_{n-2} < r a_{n-3}$

$a_{n-3} < r a_{n-4}$

— — —

$a_{m-1} < r a_{m-2}$

$a_m < r a_{m-1}$ for $n = m-1$

$\cancel{a_{m+1} < r a_m}$

Xing all above equations

Theorem # If $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l > 1$, then

$$\lim_{n\to\infty} a_n = \infty$$

Proof # $\because l > 1 \quad \therefore l-1 > 0$

we can choose a +ve number ϵ such that

$$0 < \epsilon < l-1 \quad \text{or} \quad l-\epsilon > 1$$

$\because \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l \quad \therefore \exists$ a +ven integer m such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \epsilon \qquad \forall n \geq m$$

$$\Rightarrow l-\epsilon < \frac{a_{n+1}}{a_n} < l+\epsilon \qquad \forall n \geq m$$

$$\Rightarrow \frac{a_{n+1}}{a_n} > l-\epsilon = k \longrightarrow (A) \qquad \forall n \geq m$$

putting $n = m, m+1, \dots n-1$ in (A)

$$a_{m+1} > k\, a_m$$
$$a_{m+2} > k\, a_{m+1}$$
$$a_{m+3} > k\, a_{m+2}$$
$$\text{-----------}$$
$$a_{n-2} > k\, a_{n-3}$$
$$a_{n-1} > k\, a_{n-2}$$
$$a_n > k\, a_{n-1}$$

Multiplying all above inequalities

$$a_{m+1}\, a_{m+2}\, a_{m+3} \cdots a_{n-2}\, a_{n-1}\, a_n > \varepsilon^{n-m}\, a_m\, a_{m+1}\, a_{m+2} \cdots a_{n-3}\, a_{n-2} \cdot a_{n-1}$$

$$\Rightarrow a_n > \varepsilon^{n-m}\, a_m$$

$$\Rightarrow a_n > \frac{a_m}{\varepsilon^m}\, \varepsilon^n \qquad \forall n \geq m$$

$$\Rightarrow |a_n| > \frac{|a_m|}{\varepsilon^m}\, \varepsilon^n \qquad \forall n \geq m$$

$$\because \varepsilon < 1 \quad \therefore \varepsilon^n \to \infty$$

$$\therefore \lim_{n\to\infty} a_n = \infty$$

Q# prove that for any real no ~~$\frac{x^n}{n!}$~~ x

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

Sol let $a_n = \dfrac{x^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

Q: Apply the above theorem, where a, b satisfy $0 < a < 1$, $b > 1$

Sol # (a) $\{a^n\}$ (b) $\left\{\frac{b^n}{2^n}\right\}$

(c) $\{n/b^n\}$ (d) $\left\{\frac{2^{3n}}{3^{2n}}\right\}$

Sol (a)

$$a_n = a^n \qquad 0 < a < 1$$

$$a_{n+1} = a^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{a^n} = a < 1$$

$$\Rightarrow \lim_{n\to\infty} a^n = 0$$

(b) let $a_n = \frac{b^n}{2^n}$

$$a_{n+1} = \frac{b^{n+1}}{2^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{b^{n+1}}{2^{n+1}} \times \frac{2^n}{b^n} = \frac{b}{2}$$

Q# Discuss the Convergence of following sequences where a, b are such that $0 < a <$, $b > 1$

(a) $\{n^2 a^n\}$ (b) $\{b^n/n^2\}$

(c) $\left\{\frac{b^n}{n!}\right\}$ (d) $\frac{n!}{n^n}$

Sol (a) let $a_n = n^2 a^n$

$$a_{n+1} = (n+1)^2 a^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 a^{n+1}}{n^2 a^n} = \left(\frac{n+1}{n}\right)^2 a$$

$$= (1+1/n)^2 a$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 a$$

$$= a < 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

$\Rightarrow$ $n!$ eventually increases more rapidly than a^n. This result is very useful.

(b) Let $a_n = \frac{b^n}{n^2}$

$$a_{n+1} = \frac{b^{n+1}}{(n+1)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{b^{n+1}}{(n+1)^2} \times \frac{n^2}{b^n} = \frac{n^2}{(n+1)^2} b$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^2} b$$

$$= b > 1$$

$\Rightarrow \{a_n\}$ is dgt

(c) Let $a_n = \frac{n}{b^n}$ $\quad a_{n+1} = \frac{n+1}{b^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{b^{n+1}} \times \frac{b^n}{n} = \frac{1}{b}\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{b}\left(1+\frac{1}{n}\right) = \frac{1}{b} < 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

(d) Let $a_n = \frac{n!}{n^n}$

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \frac{n^n}{(n+1)^n}.$$

$$= \left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} \overline{\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{e} < 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

Monotone Sequences

I Monotone Increasing or Non-decreasing

(a) Monotone Increasing

A sequence $\{a_n\}$ is called monotone (moving in one direction) increasing if

$$a_n \leq a_{n+1} \quad \forall n \in N$$

and is called strictly increasing if

$$a_n < a_{n+1} \quad \forall n \in N$$

(b) Monoton Decreasing or Non-Increasing Sequence

A sequence $\{a_n\}$ is called non-increasing or monotone decreasing if

$$a_n \geq a_{n+1} \quad \forall n \in N$$

and is called strictly decreasing if

$$a_n > a_{n+1} \quad \forall n \in N$$

(c) # Monotonic Sequence

A sequence $\{a_n\}$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing

(d) Strictly Monotonic Sequence.

A sequence is said to be strictly monotonic if it is either strictly monotonically increasing or strictly monotonically decreasing

Testing of Monotocity of a Sequence.

There are are several methods of testing whether a sequence $\{a_n\}$ monotone or not

(a) Difference b/w Successive Terms

Difference	Classification
$a_{n+1} - a_n > 0$	strictly increasing
$a_{n+1} - a_n < 0$	,, ,, decreasing
$a_{n+1} - a_n \geq 0$	Non-decreasing
$a_{n+1} - a_n \leq 0$	Non-increasing

(b) **By Ratio of Successive Terms #**

Ratio	Classification
$\frac{a_{n+1}}{a_n} > 1$	increasing
$\frac{a_{n+1}}{a_n} < 1$	Decreasing
$\frac{a_{n+1}}{a_n} \geq 1$	Non-decreasing
$\frac{a_{n+1}}{a_n} \leq 1$	Non-increasing

(c) If $f(x) = f(n)$ and f is differentiable then.

Derivative	Classification
$f'(x) > 0$	increasing
$f'(x) < 0$	decreasing
$f'(x) \geq 0$	Non-decreasing
$f'(x) \leq 0$	Non-increasing

(d) **Induction** Use induction on n.

Remarks # If $\{a_n\}$ is increasing, then it is bounded below by a_1 and will be bounded if it is bounded above. If $\{a_n\}$ is decreasing, it is bounded above by a_1 and will be bounded if

it is bounded below.

(2) Monotocity is very useful because it prevents the terms of a sequence from oscillating.

Eventually Monotone or Ultimately Monotone Sequence.

A sequence is eventually or ultimately monotone if $\exists$ an +ve integer m such that the sequence is monotone for all $n \geq m$. i.e sequence is montone from some term onward.

The Completeness Property of R

Every non-empty set of real numbers that has an upper bound (is bounded above) also has a Supremum in R. It is also called the least upper bound property of R.

Theorem # A monotone sequen of real nos. is convergent iff it is bounded. Further.

a) If $\{a_n\}$ is bounded monotone increasing sequence, then it converges to its supremum i.e

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in N\} = \sup_n a_n$$

(b) If $\{a_n\}$ is monotone bounded below, then it converges to its infimum i.e.

$$\lim_{n \to \infty} a_n = \inf\{a_n : n \in N\}$$
$$= \inf_n a_n$$

Proof

Necessary Condition #

Let $\{a_n\}$ be monotone convergent sequence. Then we have already proved that every cgt sequence is bound

Converse

Conversely let $\{a_n\}$ a bounded monotone sequence. Then $\exists$ nos m & M such that

$$m \le a_n \le M \quad \forall n.$$

Then $\{a_n\}$ is either increasing or decreasing

(a) Let $\{a_n\}$ be bounded increasing sequence. Then the range set $S = \{a_n : n \in N\}$ is bounded above and by least upper bound axiom of R S has l.u.b exists in R.

Let $L = \sup\{a_n : n \in N\}$

Given any $\epsilon > 0$, $L - \epsilon$ is not an upper bound of $\{a_n\}$ and there is at least one term a_m of $\{a_n\}$ greater than $L - \epsilon$ i.e.

$$L - \epsilon < a_m \longrightarrow (1)$$

otherwise $L - \epsilon$ will be an upper bound.

Since $\{a_n\}$ is monotonically increasing sequence

$$a_m \leq a_{m+1} \leq a_{m+2} \leq \cdots \longrightarrow (2)$$

By (1) & (2)

$$L - \epsilon < a_n \qquad \forall n \geq m \longrightarrow (3)$$

Also L is least upper bound of sequence

$$\therefore \quad a_n \leq L \qquad \forall n \in N \Rightarrow \forall n \geq m$$

$$\Rightarrow \quad L - a_n \geq 0 \quad \forall n \longrightarrow (4)$$

From (3) & (4)

$$0 \leq L - a_n < \epsilon \qquad \forall n \geq m$$

$$\Rightarrow \quad |L - a_n| < \epsilon \qquad \forall n \geq m$$

$$\Rightarrow \lim_{n\to\infty} a_n = L.$$

OR

Since L is supremum, therefore

$$a_n \leq L < L + \epsilon \qquad \forall n \in N$$

$$\Rightarrow \quad a_n < L + \epsilon \qquad \forall n \geq m \longrightarrow (5)$$

By (3) & (5)

$$L - \epsilon < a_n < L + \epsilon \qquad \forall n \geq m$$

$$\Rightarrow \quad |a_n - L| < \epsilon \qquad \forall n \geq m$$

$$\Rightarrow \lim_{n\to\infty} a_n = L$$

(b) Suppose the sequence $\{a_n\}$ is bounded monotically decreasing sequence.

$$M \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq m$$

Let L_1 be the g.l.b of $S = \{a_n : n \in N\}$

Given any $\epsilon > 0$, $L_1 + \epsilon > L_1$ and so $L_1 + \epsilon$ is not a lower bounde of $\{a_n\}$

$\Rightarrow \exists$ an integer m_1 such that

$$a_n \leq a_{m_1} < L_1 + \epsilon \qquad \forall n \geq m_1$$

$$\Rightarrow a_n < L_1 + \epsilon \rightarrow ③ \qquad \forall n \geq m_1$$

Also L_1 is g.l.b of $\{a_n\}$

$$a_n \geq L_1 > L_1 - \epsilon \qquad \forall n \in N$$

$$\Rightarrow a_n > L_1 - \epsilon \rightarrow ④ \quad \forall n \geq m_1$$

By ③ & ④ we have

$$|a_n - L_1| < \epsilon \qquad \forall n \geq m_1$$

$$\Rightarrow \lim_{n \to \infty} a_n = L_1$$

OR

$$L_1 \leq a_n \qquad \forall n$$

$$0 \leq a_n - L_1 \rightarrow ⑤ \; \forall n$$

by ③ & ⑤ $\quad |a_n - L_1| < \epsilon \qquad \forall n \geq m_1$

Remarks # The monotonic convergence theorem establishes the convergence of sequence without knowing the limit in advance. It also gives us a way of calculating the limit by evaluating supremum and infimum. Sometimes the supremum and infinum can not found easily but once we know that it exists, it is often possible to evaluate the limit by other methods.

Sequences defined inductly must be treated differently. If such a sequence is known to converge, then value of the limit can sometimes be determined by inductive relation.

Applications

Q# 1 # prove that the sequence defined by

$$a_n = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

is cgt

Sol clearly $a_n > 0$

$$a_{n+1} = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)$$

$$= a_n\left[1 - \frac{1}{(n+1)^2}\right] < a_n$$

$\Rightarrow \{a_n\}$ is decreasing and bounded below

It converges to l.u.b which is ≤ 3

$\therefore \lim_{n\to\infty} a_n \leq 3$

$\because a_n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}$

≥ 2

$\therefore \lim_{n\to\infty} a_n \geq 2$

Hence $2 \leq \lim a_n \leq 3$

Q#3 prove that the the sequence with general term

$a_n = 2+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

Converges

Sol # $a_n = 2+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

$a_{n+1} = 2+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{1}{(n+1)!}$

$= a_n + \frac{1}{(n+1)!} > a_n$

$\Rightarrow \{a_n\}$ is monotone increasing

using $\frac{1}{n!} < \frac{1}{2^{n-1}}$

$a_n = 2+1+\frac{1}{2!}+\frac{1}{3!}+ + \cdots + \frac{1}{n!} + \cancel{\frac{1}{(n+1)!}}$

$< 2+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$

Q: 2 # Show that the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

Converges. Also $2 \leq \lim a_n \leq 3$

Sol $$a_{n+1} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$= a_n + \frac{1}{(n+1)!} > a_n \quad \forall n.$$

$\Rightarrow \{a_n\}$ is monotone increasing. We show that $\{a_n\}$ is bounded.

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n > 1 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = 2^{n-1}$$

$$\Rightarrow \frac{1}{n!} < \frac{1}{2^{n-1}}$$

using this we have.

$$a_n = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

$$= 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}\right)$$

$$= 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - 1/2} = 1 + 2\left[1 - \left(\frac{1}{2}\right)^n\right]$$

$$< 3 \quad \forall n$$

$\Rightarrow \{a_n\}$ increases and is bounded above by 3. So $\{a_n\}$ Converges

$$< 2+1+\frac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$$

$$= 3+2\left[1-\left(\frac{1}{2}\right)^n\right] < 4.$$

Hence $\{a_n\}$ is bounded monotone sequence and so converges.

Q#4 prove that the sequence $\{a_n\}$ defined by

$$a_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$$

is dgt

Sol

$$a_{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\frac{1}{n+1}$$

$$= a_n+\frac{1}{n+1} > a_n$$

$\Rightarrow \{a_n\}$ is increasing sequence.

$$a_1 = 1$$

$$a_2 = 1+\frac{1}{2}$$

$$a_3 = a_2+\frac{1}{3}+\frac{1}{4} > \left(\frac{1}{4}+\frac{1}{4}\right) > \frac{1}{2}+a_2$$

$$a_4 = a_3+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \geq \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} > \frac{1}{2}+a_3$$

$$a_4 > 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

$$a_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) + \cdots + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2^{2-1}+1} + \frac{1}{2^2}\right) + \left(\frac{1}{2^2+1} + \frac{1}{6} + \frac{1}{7} \; \frac{1}{2^3}\right) + \left(\frac{1}{2^3+1} + \frac{1}{10} + \cdots + \frac{1}{2^4}\right) + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3}\right) + \left(\frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4}\right) + \cdots + \left(\frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}\right).$$

$$= 1 + \frac{1}{2} + \Big(\underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{(n-1)\text{ terms}}\Big).$$

$$= 1 + \frac{n}{2}$$

$$a_{2^n} > 1 + \frac{n}{2}$$

$\Rightarrow$ Sequence a_n is unbounded and hence divergent.

Q:5# prove that the sequence

$$x_1 = \frac{x_0}{a+x_0}, \quad x_2 = \frac{x_1}{a+x_1}, \dots, x_n = \frac{x_{n-1}}{a+x_{n-1}}, \dots$$

$a > 1$, $x_0 > 0$ Converges

Sol

$$x_n = \frac{x_{n-1}}{a+x_{n-1}} < x_{n-1}$$

$\Rightarrow x_n < x_{n-1}$

$\Rightarrow$ Sequence $\{x_n\}$ is decreasing

2ndly since $a > 1$, $x_0 > 0$, Therefore all of the terms are +ve which means that sequence is bounded below. Thus the sequence is monotone and bounded. Hence it is cgt.

Q#6 prove that the sequence with general term

$$a_n = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1} + \dots + \frac{1}{5^n+1}$$

i.e.

$$a_1 = \frac{1}{5+1} \qquad a_2 = \frac{1}{5+1} + \frac{1}{5^2+1}$$

$$a_3 = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1}$$ is cgt

Sol

$$a_{n+1} = \frac{1}{5+1} + \frac{1}{5^2+1} + \cdots + \frac{1}{5^n+1} + \frac{1}{5^{n+1}+1}$$

$$= a_n + \frac{1}{5^{n+1}+1}$$

$\Rightarrow a_{n+1} > a_n$

$\Rightarrow \{a_n\}$ is increasing

Also since $\frac{1}{5^n+1} < \frac{1}{5^n} \quad \forall n$.

We have.

$$a_n = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1} + \cdots + \frac{1}{5^n+1}$$

$$< \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots + \frac{1}{5^n}$$

$$= \frac{\frac{1}{5} - \frac{1}{5^{n+1}}}{1 - 1/5} = \frac{5}{4}\left[\frac{1}{5} - \frac{1}{5^{n+1}}\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{5^n}\right] < \frac{1}{4} \quad \forall n$$

Hence the sequence is convergent.

R.G Q#7* prove that the sequence

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

is convergent and its limit lies between $\frac{1}{2}$ and 1

Sol

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

$$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}.$$

$$a_{n+1} - a_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}$$

$$> \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{2}{2n+2} - \frac{1}{n+1} = 0$$

$a_{n+1} > a_n$

$\Rightarrow \{a_n\}$ is monotonically increasing

Also

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$$

$$< \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} + \cdots \frac{1}{n+1}$$

$$= \frac{n}{n+1} < 1 \quad \forall n$$

$\Rightarrow \{a_n\}$ is bounded above by 1

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n}$$

$$> \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

$a_n > \frac{1}{2}.$

Thus $\frac{1}{2} < a_n < 1 \quad \forall n.$

Hence $\frac{1}{2} \le \lim_{n\to\infty} a_n \le 1$

Q: 8 R.G.B # Gaskil Ex. Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ $\forall n \in N$. prove that $\{x_n\}$ is increasing and bounded and hence converges:

Sol we not that

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k} \quad \forall k \ge 2 \longrightarrow (1)$$

we will use this fact

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

$$x_{n+1} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2}$$

$$= x_n + \frac{1}{(n+1)^2} > x_n \quad \forall n.$$

$\Rightarrow \{x_n\}$ is increasing

Also by using (1)

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

$$< \frac{1}{1^2} + \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right)$$

$$= \frac{1}{1^2} + \frac{1}{1} - \frac{1}{n} = 2 - \frac{1}{n} < n \quad \forall n$$

49
WA

$\Rightarrow \{x_n\}$ is bounded a ... ing. he
is cgt

Calculation of S

Q#(9) # (a) A sequence ... ined by
$a_1 = 2$, $a_{n+1} = \frac{1}{2}$ ...)
Prove that $\{a_n\}$ converg ... its limi
RG.B (b) # Let $a > 0$, we ... sequenc
which converges to $\sqrt{a}$
Let $s_1 > 0$ be an ar ... nd.
$s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s}\right.$ ... $n \in N$
Prove that $\{s_n\}$ conver ... find its li
Note It is general case of

Sol # (a) $a_1 = 2.$
$a_2 = \frac{1}{2}(2$ ... $3/2 = 1.5$
$a_3 = \frac{1}{2}(3/2$ ... $= 1.4167$
$a_4 = \frac{1}{2}(\frac{1}{12}$ ... $= \frac{577}{408} = 1.4$

Casual reader may ... t $\{a_n\}$ is
decreasing.

OR

$\Rightarrow \{x_n\}$ is bounded and increasing. hence is cgt

Calculation of Square root

Q#(9) # (a) A sequence $\{a_n\}$ is defined by

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

prove that $\{a_n\}$ converges and find its limit

R.G.B (b) # Let $a > 0$, we construct a sequence which converges to $\sqrt{a}$

Let $s_1 > 0$ be an arbitrary and

$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right) \quad \forall n \in N$$

prove that $\{s_n\}$ converges ~~and~~ find its limit.

Note It is general case of (a)

Sol # (a) $a_1 = 2.$

$$a_2 = \frac{1}{2}\left(2 + \frac{2}{2}\right) = 3/2 = 1.5$$

$$a_3 = \frac{1}{2}\left(3/2 + \frac{2}{3/2}\right) = 1.4167$$

$$a_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{\frac{17}{12}}\right) = \frac{577}{408} = 1.4142$$

Casual reader may deduce that $\{a_n\}$ is decreasing.

OR

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$$

$$a_n^2 = \frac{1}{4}\left[a_{n-1} + \frac{2}{a_{n-1}}\right]^2$$

$$= \frac{1}{4}\left[\left(a_{n-1} - \frac{2}{a_{n-1}}\right)^2 + 8\right]$$

$$= \frac{1}{4}\left(a_{n-1} - \frac{2}{a_{n-1}}\right)^2 + 2 \geq 2 \quad \forall n \geq 2.$$

$$\Rightarrow a_n^2 \geq 2.$$

$$\Rightarrow a_n \geq \frac{2}{a_n} \qquad \forall n \geq 2$$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \leq \frac{1}{2}(a_n + a_n) = a_n$$

$$a_{n+1} \leq a_n \quad \forall n \geq 2.$$

$\Rightarrow \{a_n\}$ is ultimately decreasing.

OR

By $\quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$

$$2a_{n+1}a_n = a_n^2 + 2$$

$$\Rightarrow a_n^2 - (2a_{n+1})a_n + 2 = 0$$

$\Rightarrow a_n$ satisfies the quadratic equation.

$\because a_1 = 2 \quad \therefore$ This quadratic equation has

has a real root.

$\Rightarrow$ Disc of it must be non-negative.

$$\Rightarrow \quad 4a_{n+1}^2 - 4(2) \geq 0 \qquad \forall n \geq 1$$

$$\Rightarrow \quad a_{n+1}^2 - 2 \geq 0 \qquad \forall n \geq 1$$

$$\Rightarrow \quad a_{n+1}^2 \geq 2 \qquad \forall n \geq 1$$

$$\Rightarrow \quad a_n^2 \geq 2 \qquad \forall n \geq 2.$$

Now

$$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

$$= a_n - \frac{1}{2}\left(\frac{a_n^2 + 2}{a_n}\right)$$

$$= \frac{2a_n^2 - a_n^2 - 2}{2a_n} \qquad \forall n \geq 2$$

$$= \frac{1}{2}\frac{(a_n^2 - 2)}{a_n} \geq 0 \qquad \because a_n^2 \geq 2, \; a_n > 0 \; \forall n.$$

$$\Rightarrow \quad a_{n+1} \leq a_n \qquad \forall n \geq 2$$

$\Rightarrow \{a_n\}$ is decreasing and is bounded below by 0

$\Rightarrow \lim_{n\to\infty} a_n$ exists.

let $\lim_{n\to\infty} a_n = l$.

..., $\lim_{n\to\infty} a_n$ is a the root of $x^2 - 2x - 1 = 0$

91

$\therefore a_n > 0 \quad \forall n$

$\therefore l \geq 0$

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} a_n + \frac{2}{\lim_{n\to\infty} a_n}\right)$$

$$l = \frac{1}{2}\left(l + \frac{2}{l}\right) = \frac{l}{2} + \frac{1}{l}.$$

$$2l^2 = l^2 + 2.$$

$$l^2 = 2 \Rightarrow l = \pm\sqrt{2}$$

$\therefore l \geq 0$

$\therefore$ only possibility is $l = \sqrt{2} \approx 1.414214$

(b) $$s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$$

$$s_n = \frac{1}{2}\left(s_{n-1} + \frac{a}{s_{n-1}}\right)$$

$$\Rightarrow s_n^2 = \frac{1}{4}\left(s_{n-1} + \frac{a}{s_{n-1}}\right)^2$$

$$= \frac{1}{4}\left[\left(s_{n-1} - \frac{a}{s_{n-1}}\right)^2 + 4a\right]$$

$$= \frac{1}{4}\left(s_{n-1} - \frac{a}{s_{n-1}}\right)^2 + a \geq a \quad \forall n \geq 2$$

$$\Rightarrow s_n^2 \geq a \Rightarrow \frac{a}{s_n} \leq s_n \quad \forall n \geq 2$$

$$\Rightarrow s_n \geq \frac{a}{s_n}.$$

$\therefore\ s_n > 0 \quad \forall n$

$l \geq 0$

$s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$

$\lim_{n\to\infty} s_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} s_n + \frac{a}{\lim_{n\to\infty} s_n}\right)$

$l = \frac{1}{2}\left(l + \frac{a}{l}\right)$

$2l^2 = l^2 + a.$

$l^2 = a.$

$l = \sqrt{a}$

<u>Note</u> For the purpose of calculation, it is often important to have an estimate of how rapidly the sequence s_n converges to $\sqrt{a}$.

$\therefore\ \lim_{n\to\infty} s_n = \inf_n s_n = \sqrt{a}$

$\therefore\ \sqrt{a} \leq s_n \quad \forall n \geq 2.$

$\Rightarrow\ a \leq s_n^2 \quad \forall n \geq 2.$

$\Rightarrow\ \frac{a}{s_n} \leq s_n. \quad \forall n \geq 2$ or $\frac{a}{s_n} \leq \sqrt{a} \leq s_n$

$0 \leq s_n - \sqrt{a} \leq s_n - \frac{a}{s_n} \quad \because \frac{a}{s_n} \leq \sqrt{a}$

$= \frac{s_n^2 - a}{s_n} \quad \forall n \geq 2.$

$0 \leq \frac{s_n^2 - a}{s_n}$ Using this inequality we can calculate $\sqrt{a}$ to any desired degree of accuracy.

$l = 2 + \frac{1}{l}$

Thus $\lim_{n\to\infty} x_n$ is a +ve root of $x^2 - 2x - 1 = 0$

$$S_{n+1} = \frac{1}{2}\left(S_n + \frac{a}{S_n}\right) \leq \frac{1}{2}(S_n + S_n) \quad \forall n \geq 2$$

$$S_{n+1} \leq S_n \quad \forall n \geq 2$$

$\Rightarrow \{S_n\}$ is decreasing

OR

$$S_{n+1} = \frac{1}{2}\left(S_n + \frac{a}{S_n}\right)$$

$\Rightarrow S_n^2 - (2S_{n+1})S_n + a = 0$

$\Rightarrow S_n$ satisfies quadratic equation & $S_1 > 0$

$\Rightarrow$ This equation has real roots

$\Rightarrow$ Disc ≥ 0

$$4S_{n+1}^2 - 4a \geq 0$$

$$S_{n+1}^2 \geq a \qquad \forall n \geq 1$$

or

$$S_n^2 \geq a \qquad \forall n \geq 2$$

$$S_n - S_{n+1} = \frac{1}{2}\left(\frac{S_n^2 - a}{S_n}\right) \geq 0 \quad \forall n \geq 2$$

$\because S_n^2 \geq a$, $S_n > 0$ $\forall n$

$\Rightarrow S_{n+1} \leq S_n \quad \forall n \geq 2$

$\Rightarrow \{S_n\}$ is ultimately decreasing.

Also $S_n \geq 0 \quad \forall n$.

Thus $\{S_n\}$ monotone bounded and hence is cgt. Let $\lim_{n\to\infty} S_n = l$.

94

Exercise Show that the sequence defined by

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right) \quad n \geq 1, a_1 > 0$$

Converges to 3

Q# 10 # (G.R.B (example)) prove that the sequence defined by

$$x_1 = 2 \qquad x_{n+1} = 2 + \frac{1}{x_n} \quad \forall n \in N$$

is convergent and converges to a +ve root of equation $x^2 - 2x - 1 = 0$

Sol #

$$x_{n+1} = 2 + \frac{1}{x_n} > 2 \quad \forall n \geq 1$$

$$\& \; x_1 = 2$$

$$\Rightarrow x_n \geq 2 \qquad \forall n \geq 1$$

Also

$$x_1 = 2$$

$$x_2 = 2 + \frac{1}{x_1} = 2 + \frac{1}{2} = 5/2 = 2.5$$

$$x_3 = 2 + \frac{1}{x_2} = 2 + \frac{2}{5} = \frac{12}{5} = 2.4$$

$$x_4 = 2 + \frac{1}{x_3} = 2 + \frac{5}{12} = \frac{29}{12}$$

$$x_5 = 2 + \frac{12}{29} = \frac{70}{29}$$

We note that sequence ultimately decrease.

$\Rightarrow \{x_n\}$ bounded monotone and hence cgt.

let $\lim_{n\to\infty} x_n = l$. Then

$$l = 2 + \frac{1}{l} \Rightarrow l^2 - 2l - 1 = 0$$

Thus $\lim_{n\to\infty} x_n$ is a +ve root of $x^2 - 2x - 1 = 0$

and $\lim_{n\to\infty} x_n = 1+\sqrt{2}$

Q#11 (R.G.B) Let $\{y_n\}$ be defined by

$y_1 = 1$ $\quad y_{n+1} = \frac{1}{4}(2y_n + 3)$ for $n \geq 1$

Show that $\lim_{n\to\infty} y_n = 3/2$

Sol By direct calculation.

$$y_2 = 5/4$$

Hence $y_1 < y_2 < 2$.

We show by induction that $y_n < 2$ $\forall n \in N$

It is true for $n = 1, 2$.

Let $y_k < 2$ for some $k \in N$, $k > 2$

Then $y_{k+1} = \frac{1}{4}(2y_k + 3) < \frac{1}{4}(2\cdot 2 + 3) = 7/4 < 2$

$\Rightarrow y_{k+1} < 2 \Rightarrow y_n < 2$ $\forall n \in N$

We show by induction that $y_n < y_{n+1}$.

For $n = 1$ $\quad y_2 = 5/4 = 1.25 > 1 = y_1$

$\Rightarrow y_1 < y_2$

Let $y_k < y_{k+1}$ for some $k \in N$

Then. $2y_k < 2y_{k+1}$

$2y_k + 3 < 2y_{k+1} + 3$

$y_{k+1} = \frac{1}{4}(2y_k + 3) < \frac{1}{4}(2y_{k+1} + 3) = y_{k+2}$.

Thus $z_n < 2 \quad \forall n \geq 2$

Also $z_1 = 1$

$\Rightarrow 1 \leq z_n < 2 \quad \forall n \in N$

$\Rightarrow \{z_n\}$ is bounded.

<u>OR</u>

$z_2 = \sqrt{2} < 2$

$z_3 = \sqrt{2z_2} < \sqrt{2.2} = 2$

$z_4 = \sqrt{2z_3} < \sqrt{2.2} = 2$

$z_5 = \sqrt{2z_4} < \sqrt{2.2} = 2$

$\vdots$

$\Rightarrow z_n < 2 \quad \forall n \geq 2.$

<u>OR</u>

$z_1 = 1$

$z_2 = \sqrt{2z_1} = \sqrt{2} = 2^{1/2}$

$z_3 = \sqrt{2z_2} = \sqrt{2.\sqrt{2}} = 2^{1/2+1/4} = 2^{\frac{1}{2}+\frac{1}{2^2}}$

$z_4 = \sqrt{2z_3} = \left(2 . 2^{\frac{1}{2}+\frac{1}{2^2}}\right)^{1/2} = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}$

$\vdots$

$z_n = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}}$

$z_{n+1} = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}} < 2$

$= z_n \cdot 2^{\frac{1}{2^n}} \geq z_n$ Also

Thus $y_k < y_{k+1} \Rightarrow y_{k+1} < y_{k+2}$

$\Rightarrow$ $y_n < y_{n+1}$ $\forall n \in N$.

$\Rightarrow \{y_n\}$ is increasing and bounded above by 2

$\Rightarrow \{y_n\}$ is cgt.

Let $\lim\limits_{n\to\infty} y_n = l = \lim\limits_{n\to\infty} y_{n+1}$

Then from $y_{n+1} = \frac{1}{4}(2y_n + 3)$

$$l = \frac{1}{4}(2l+3)$$

$$4l = 2l + 3$$

$$2l = 3 \Rightarrow l = 3/2 = 1.5$$

Q # 12 (R.G.B) Show that the sequence of real numbers defined by

$$z_1 = 1 \qquad z_{n+1} = \sqrt{2z_n} \text{ Converges}$$

and $\lim\limits_{n\to\infty} z_n = 2$.

Sol $z_1 = 1 \qquad z_2 = \sqrt{2} < 2$

$\Rightarrow z_1 < z_2 < 2 \longrightarrow (1)$

For $n = 2$, we have

$$z_2 = \sqrt{2} < 2.$$

Let $z_k < 2$ for $k > 2$, $k \in N$.

Then $z_{k+1} = \sqrt{2z_k} < \sqrt{2.2} = 2.$

Q: 12

Show that the sequence $\{a_n\}$ defined by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$ Converges to 2

Sol

$$a_1 = \sqrt{2} < 2.$$

$$a_2 = \sqrt{2a_1} < \sqrt{2 \cdot 2} < 2$$

$$a_3 = \sqrt{2a_2} < \sqrt{2 \cdot 2} < 2.$$

$$\cdots\cdots$$

$$a_n < 2 \quad \forall n.$$

Also

$$a_1 = \sqrt{2}$$

$$a_2 = \sqrt{2\sqrt{2}} > \sqrt{2} = a_1.$$

$$a_2 > a_1.$$

Let $a_{k+1} > a_k \rightarrow (1)$ for some $k \in N$

Then $a_{k+2} = \sqrt{2a_{k+1}}$ & $a_{k+1} = \sqrt{2a_k}$

from (1)

$$2a_{k+1} > 2a_k$$

$$\Rightarrow \sqrt{2a_{k+1}} > \sqrt{2a_k}$$

$$a_{k+2} > a_{k+1}$$

$$a_{k+1} > a_k \Rightarrow a_{k+2} > a_{k+1}$$

$\Rightarrow \{z_n\}$ is bounded and increasing.

OR

Increasing fact can be proved by induction as

$$z_1 < z_2 \quad \rightarrow ①$$

We are to prove that $\quad \forall n.$

$$z_n \leq z_{n+1}$$

For $n=1$ inequality is true from ①

Let it be true for $n = k$. i.e.

$$z_k \leq z_{k+1} \quad \rightarrow ②$$

Now $z_{k+1} = \sqrt{2 z_k} < \sqrt{2 z_{k+1}} = z_{k+2}$ $\quad$ by ②

$\Rightarrow z_{k+1} < z_{k+2}$

Thus $z_k < z_{k+1} \Rightarrow z_{k+1} < z_{k+2}$.

$\Rightarrow \quad z_n < z_{n+1} \quad \forall n.$

$\Rightarrow \{z_n\}$ increasing sequence & bounded.

So $\{z_n\}$ is cgt. Let $\lim_{n\to\infty} \{z_n\} = l$.

Then. $\quad z_{n+1} = \sqrt{2 z_n}$

$\Rightarrow \quad l = \sqrt{2l}.$

$l^2 = 2l. \Rightarrow \quad l^2 - 2l = 0$

$l(l-2) = 0$

$\Rightarrow l = 0 \quad l = 2. \quad$ But $1 \leq l \leq 2$

$\Rightarrow \boxed{l = 2}$

100

2

$a_1 =$

$a_2 = \sqrt{2\sqrt{2}} = 2^{1/2} \cdot 2^{1/4} = 2^{\frac{1}{2}+\frac{1}{2^2}}$

$a_3 = \sqrt{\;\;} \left(2 \cdot 2^{1/2+\frac{1}{4}}\right)^{1/2}$

$= 2 \quad + \frac{1}{8} \quad 2^{\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}}$

$= 2$

$\frac{1}{2^2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}$

$a_n =$

$\frac{1}{2^2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}}$

$a_{n+1} =$

$a_{n+1} = \quad 2^{\frac{1}{2^{n+1}}} \geq a_n \ \forall n$

$$\boxed{\begin{aligned} &\because 2^{1/n} > 1 \quad \forall n > 1 \\ &\because 2^{\frac{1}{2^{n+1}}} > 1 \\ &\Rightarrow a_n \cdot 2^{\frac{1}{2^{n+1}}} > a_n \\ &a_{n+1} > a_n \end{aligned}}$$

$\Rightarrow \{a_n\}$ reasing

Again. $1/2 + \quad \cdots + \frac{1}{2^n}$

$n = 2 \quad < 2^1 \ \forall n$

$\Rightarrow a_n < 2 \quad n$

$\Rightarrow \{a_n\}$ b ed monotone and

hence vergent. Let $\lim_{n\to\infty} a_n = l$

100

OR

$a_1 = \sqrt{2}$

$a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}} = 2^{1/2} \cdot 2^{1/4} = 2^{\frac{1}{2}+\frac{1}{2^2}}$

$a_3 = \sqrt{2a_2} = \left(2 \cdot 2^{1/2+\frac{1}{4}}\right)^{1/2}$

$= 2^{1/2+\frac{1}{4}+\frac{1}{8}} = 2^{\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}}$

$a_n = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}$

$a_{n+1} = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}}}$

$a_{n+1} = a_n \cdot 2^{\frac{1}{2^{n+1}}} \geq a_n \ \forall n$

$\boxed{\begin{array}{l} \because 2^{1/n} > 1 \quad \forall n > 1 \\ \therefore 2^{\frac{1}{2^{n+1}}} > 1 \\ \Rightarrow a_n \cdot 2^{\frac{1}{2^{n+1}}} > a_n \\ a_{n+1} > a_n \end{array}}$

$\Rightarrow \{a_n\}$ is increasing

Again.

$a_n = 2^{1/2+\frac{1}{2^2}+\cdots+\frac{1}{2^n}} < 2^1 \ \forall n$

$\Rightarrow a_n < 2 \quad \forall n$

$\Rightarrow \{a_n\}$ bounded monotone and hence is Converget. Let $\lim_{n\to\infty} a_n = l$

...n $\lim_{n\to}$ ... l

a_{n+1} ... $\sqrt{2a_n}$

$\Rightarrow a_{n+}^2$ ... $2a_n$

$\Rightarrow (\lim_{n\to\infty} a$ ... $2\lim_{n\to\infty} a$

l ... $2l \Rightarrow$

$l = 0$ & ...

...ut $a_n >$ ... $\forall n$.

$l = 2$ ... $\lim_{n\to\infty} a_n$

Show that ... sequence

...ed by ... $= \sqrt{3 s_n}$

...es to 3

s ... 1

$s_2 =$ ... $s_1 = \sqrt{3}$

$s_3 =$ ... $= \sqrt{3 \cdot 3^{1/}}$

$s_4 =$ ... $+ \frac{1}{2^2} + \frac{1}{2^3}$

$s_n = 3^{\frac{1}{2} + \dots + \frac{1}{2^3} + \dots +}$

$\dots = 3^{\frac{1}{2} + \frac{1}{2} \dots + \dots + \frac{1}{2^n}}$

$= a_n$ ... $> a_n$

$\Rightarrow \boxed{l =}$

Then $\lim\limits_{n\to\infty} a_{n+1} = l$.

$a_{n+1} = \sqrt{2a_n}$

$\Rightarrow a_{n+1}^2 = 2a_n$

$\Rightarrow \left(\lim\limits_{n\to\infty} a_{n+1}\right)^2 = 2\lim\limits_{n\to\infty} a_n$

$l^2 = 2l \Rightarrow l(l-2) = 0$

$\Rightarrow l = 0$ & $l = 2$

But $a_n > 0 \quad \forall n$.

$\Rightarrow l = 2 \Rightarrow \lim\limits_{n\to\infty} a_n = 2$.

Q:13 Show that the sequence $\{s_n\}$ defined by $s_{n+1} = \sqrt{3s_n}$, $s_1 = 1$ Converges to 3

Sol

$s_1 = 1$

$s_2 = \sqrt{3s_1} = \sqrt{3} = 3^{1/2}$.

$s_3 = \sqrt{3s_2} = \sqrt{3\cdot 3^{1/2}} = 3^{\frac{1}{2}+\frac{1}{2^2}}$

$s_4 = 3^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}$

- - ———— ————

———————————————

$s_n = 3^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}}$

$\Rightarrow s_{n+1} = 3^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}}$

$= a_n \cdot 3^{\frac{1}{2^n}} > a_n \quad \forall n$.

$\Rightarrow \boxed{l = 2}$

$\underline{\text{Sol}}$ # [redacted] $\underline{\underline{1\cdot 3}}$ [redacted]

$x_n =$ [redacted]

$= \sqrt{2+}$ [redacted]

$\therefore$

Suppo [redacted] $k > x$ [redacted]

$\Rightarrow$ [redacted] $k > 2+$ [redacted]

$\Rightarrow$ [redacted] $k > \sqrt{2}$ [redacted]

$\Rightarrow$ [redacted] $1 > x$ [redacted] $k+1-1$

induction.

Thus [redacted] $> x_n \ \forall$ [redacted]

Now [redacted] $= 1 < 2$ [redacted] 2

x [redacted] $x_1 < \sqrt{2}$ [redacted] 2

x [redacted] $x_2 < \sqrt{2}$ [redacted] 2

[redacted] $3+2 < \sqrt{2}$ [redacted] 2

[redacted] $2 \quad \forall$ [redacted]

$\underline{\underline{\text{OR}}}$

$x_1 <$ [redacted] OR

By [redacted] < 2 [redacted] $2\sqrt{2+x_2}$

Let [redacted] < 4 [redacted] $< \sqrt{2+2}$

Then [redacted] < 2 [redacted] we

$\Rightarrow$ [redacted] $h < 2$ [redacted] $n < 2 \ \forall$

$\Rightarrow$ [redacted] $1 < 2$ [redacted]

$< 17+$ [redacted]

Sol #

$x_1 = 1$ 1.3

$x_n = \sqrt{2 + x_{n-1}}$

$x_2 = \sqrt{2 + x_1} = \sqrt{2+1} = \sqrt{3} > 1 = x_1$

$\therefore \quad x_2 > x_1$

Suppose that $x_k > x_{k-1}$ for $k \in N$

$\Rightarrow \quad 2 + x_k > 2 + x_{k-1}$

$\Rightarrow \quad \sqrt{2 + x_k} > \sqrt{2 + x_{k-1}}$

$\Rightarrow \quad x_{k+1} > x_k = x_{k+1-1}$

Thus $\quad x_{n+1} > x_n \quad \forall n$ by induction.

Now $\quad x_1 = 1 < 2$

$x_2 = \sqrt{2 + x_1} < \sqrt{2+2} < 2$

$x_3 = \sqrt{2 + x_2} < \sqrt{2+2} < 2$

$x_4 = \sqrt{x_3 + 2} < \sqrt{2+2} < 2$

— — — —

$x_n < 2 \quad \forall n.$

OR

By induction $\quad x_1 < 2$

Let $\quad x_k < 2$ OR $x_{k+1} = \sqrt{2 + x_k} < \sqrt{2+2} = 2$

Then $\quad 2 + x_k < 4$

$\Rightarrow \quad \sqrt{2 + x_k} < 2$ Hence

$\Rightarrow \quad x_{k+1} < 2.$ $x_n < 2 \quad \forall n$

$/ 17 + a_{k-1}$

Also $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}} < 3$ $\quad 1$

$S_n = 3$

$\because \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}} < 1.$

$\Rightarrow S_n < 3 \quad \forall n.$

Hence $\{a_n\}$ is bounded and converges.

$S_n \geq 1 \quad \forall n.$

Let $\lim_{n\to\infty} S_n = l$ $\qquad \therefore \lim_{n\to\infty} S_n \geq 1$.

$S_{n+1} = \sqrt{3}\sqrt{S_n}$

$S_{n+1}^2 = 3S_n$

$\Rightarrow \left(\lim_{n\to\infty} S_{n+1}\right)^2 = 3\lim_{n\to\infty} S_n$

$l^2 = 3l \Rightarrow l(l-3) = 0$

$\Rightarrow l = 0 \text{ \& } l = 3$

$\because \lim_{n\to\infty} S_n \geq 1 \Rightarrow \lim_{n\to\infty} S_n = 3$

Q# 14 If $S_{n+1} = \sqrt{7S_n}$ $S_1 = 1$, prove that $\{S_n\}$ is Convergent. What is its limit.

try yourself.

Thus $\{x_n\}$ bounded monotone sequence.
and

$$1 \leq a_n < 2.$$

$$\Rightarrow \quad 1 \leq \lim_{n\to\infty} a_n \leq 2$$

Let

$$\lim_{n\to\infty} x_n = l.$$

$$x_n = \sqrt{2 + x_{n-1}} \Rightarrow x_n^2 = 2 + x_{n-1}$$

$$\Rightarrow (\lim x_n)^2 = 2 + \lim x_{n-1}$$

$$l^2 = 2 + l.$$

$$\Rightarrow l^2 - l - 2 = 0$$

$$\Rightarrow (l-2)(l+1) = 0$$

$$\Rightarrow l = 2 \qquad l = -1$$

$$\because x_n > 0 \ \forall n \Rightarrow l = 2$$

Q.16 prove that the sequence $\{a_n\}$ defined by $a_1 = \sqrt{7}$ & $a_{n+1} = \sqrt{7 + a_n}$ Converges to the tue square root of $x^2 - x - 7 = 0$

Sol

$$a_1 = \sqrt{7} \qquad a_{n+1} = \sqrt{7 + a_n}$$

$$a_2 = \sqrt{7 + a_1} = \sqrt{7 + \sqrt{7}} > \sqrt{7} = a_1$$

Let $a_k > a_{k-1}$ for some $k \in N$

$$7 + a_k > 7 + a_{k-1}$$

$$\Rightarrow \sqrt{7 + a_k} > \sqrt{7 + a_{k-1}} \Rightarrow a_{k+1} > a_k.$$

By Mathematical induction.

$\Rightarrow$ $a_{n+1} > a_n \quad \forall n.$

$\{a_n\}$ is monotonically increasing

Now $a_1 = \sqrt{7} < 7$

Let $a_k < 7$ for some $k \in N$

$$7 + a_k < 7 + 7 = 14$$

$$\sqrt{7 + a_k} < \sqrt{14} < \sqrt{49} = 7$$

$\Rightarrow a_{k+1} < 7$

$\Rightarrow$ By mathematical induction.

$$a_n < 7 \quad \forall n.$$

$\Rightarrow$ $\{a_n\}$ is bounded above

Since $\{a_n\}$ is monotonically increasing and bounded above, it is cgt.

Let $\lim\limits_{n \to \infty} a_n = l$

Now $a_{n+1} = \sqrt{7 + a_n} \Rightarrow a_{n+1}^2 = 7 + a_n$

$\Rightarrow l^2 = 7 + l \Rightarrow l^2 - l - 7 = 0$

$\because a_n > 0 \ \forall n.$

$\therefore$ $\{a_n\}$ Converges to a +ve root of $x^2 - x - 7 = 0$

$$l = \frac{1 \pm \sqrt{1 + 28}}{2} = \frac{1 \pm \sqrt{29}}{2}$$

But $\frac{1 - \sqrt{29}}{2} < 0$ so $l = \frac{1 + \sqrt{29}}{2}$.

By Mathematical induction.

$a_{n+1} > a_n \quad \forall n.$

$\Rightarrow$ $\{a_n\}$ is monotonically increasing

Now $\quad a_1 = \sqrt{7} < 7$

Let $\quad a_k < 7 \quad$ for some $k \in N$

$$7 + a_k < 7+7 = 14.$$

$$\sqrt{7+a_k} < \sqrt{14} < \sqrt{49} = 7$$

$\Rightarrow \quad a_{k+1} < 7$

$\Rightarrow$ By mathematical induction.

$$a_n < 7 \quad \forall n.$$

$\{a_n\}$ is bounded above

ice $\{a_n\}$ is monotonically increasing a

nded above, it is cgt.

et $\quad \lim\limits_{n\to\infty} a_n = l$

w $\quad a_{n+1} = \sqrt{7+a_n} \Rightarrow a_{n+1}^2$

$$l^2 = 7 + l \Rightarrow l^2 - l - 7 =$$

$a_n > 0 \ \forall n.$

$\{a_n\}$ Converges to a +ve root of

$$= \frac{1 \pm \sqrt{1+28}}{2} = \frac{1 \pm \sqrt{2}}{2}$$

$\frac{1-\sqrt{29}}{2} < 0 \quad$ So $l = \frac{1+\sqrt{29}}{2}$

Thus

$x_1 < \alpha$

(or)

$x_1 > \alpha \quad , \quad x_1 > -\beta$

$\Rightarrow x_1 > \alpha$

Thus $\{x_n\}$ is increasing or decreasing according as $x_1 = \sqrt{a}$ is less or greater than the +ve root of the equation.

$$x^2 - x - a = 0 \text{ or } x^2 = x + a$$

Thus

$$x_1 > \alpha \Rightarrow (2) \text{ i.e.}$$

$$x_1^2 - x_1 - a > 0$$

$$x_1^2 > x_1 + a$$

$$x_1 > \sqrt{x_1 + a}$$

$$\sqrt{x_1 + a} < x_1 \Rightarrow x_2 < x_1$$

$\Rightarrow \{x_n\}$ is monotonically ~~decreasing~~ decreasing

Now $x_n^2 = x_{n-1} + a > x_n + a \quad \because x_n < x_{n-1}$

$$x_n^2 - x_n - a > 0$$

from (2) $\Rightarrow x_n > \alpha \ \forall n$

OR

$x_1 > \alpha$.

Let $x_k > \alpha$ for some $k \in N$.

$\sqrt{a + x_k} > \sqrt{a + \alpha} \quad \because x_k > \alpha$.

$x_{k+1} > \sqrt{\alpha^2 + \alpha} > \alpha$.

$x_1 > \alpha$

$x_1^2 > \alpha^2$

$a > \alpha^2$

Thus $x_n > \alpha \ \forall n$ by induction.

Q #17 (R.G.B)

If $x_1 = \sqrt{a}$ $(a > 0)$ & $x_{n+1} = \sqrt{a + x_n}$ $\forall n \in N$ show that $\{x_n\}$ converges and find its limit.

Sol
$$x_{n+1}^2 - x_n^2 = (a + x_n) - (a + x_{n-1})$$
$$= x_n - x_{n-1}$$

Thus we note that

$x_n > x_{n-1} \Rightarrow x_{n+1} > x_n$.

$x_n > x_{n-1} \Rightarrow x_{n+1} < x_n$.

and $x_n < x_{n-1} \Rightarrow$

Thus $\{x_n\}$ is a monotone sequence

Also $\{x_n\}$ is an increasing or decreasing according

as $x_2 > x_1$ or $x_2 < x_1$

x_1

$\sqrt{a + x_1} > x_1^2$

$\Rightarrow a + x_1 > \cancel{<} 0 \rightarrow ①$

$x_1^2 - x_1 - a$

$\sqrt{a + x_1} < x_1$

$a + x_1 < x_1^2$

$x_1^2 - x_1 - a > 0 \rightarrow ②$

Now product of root of $x_1^2 - x_1 - a = 0$

$= -\frac{a}{1} < 0$

$\Rightarrow$ one of roots is $-$ve and other is +ve.

$x_1 = \frac{1 \pm \sqrt{4a+1}}{2}$

$\alpha = \frac{1 + \sqrt{4a+1}}{2} > 0$

$-\beta = \frac{1 - \sqrt{4a+1}}{2} < 0$ $\beta >$

① $\Rightarrow (x_1 - \alpha)(x_1 + \beta) < 0$

$\because x_1 > 0$
$x_1 + \beta \not< 0$

$\Rightarrow x_1 - \alpha < 0$

$x_1 < \alpha = \frac{1 + \sqrt{4a+1}}{2}$

② $\Rightarrow (x_1 - \alpha)(x_1 + \beta) > 0$

$\Rightarrow x_1 - \alpha > 0$ & $x_1 + \beta > 0$

Thus $\{x_n\}$ is monotonically decreasing sequence which is bounded below. Let $\lim_{n\to\infty} x_n = l$.
Hence $\lim_{n\to\infty} a_n$ exists.
We have $l \geq \alpha$.
Now

$$x_{n+1} = \sqrt{a + x_n}$$

$$\Rightarrow l^2 = l + a$$

$$l^2 - l - a = 0$$

$$l = \frac{1 \pm \sqrt{1+4a}}{2} \Rightarrow l = \frac{1+\sqrt{4a+1}}{2} = \alpha > 0$$

Similarly $x_1 < \alpha$ from ①

$$\Rightarrow x_1^2 - x_1 - a < 0$$

$$x_1 < \sqrt{x_1 + a} = x_2$$

$$x_1 < x_2$$

$\Rightarrow \{x_n\}$ is montonically increasing

Now $x_n^2 = x_{n-1} + a < x_n + a$.

$$x_n^2 - x_n - a < 0$$

$\Rightarrow x_n < \alpha \quad \forall n$ by ①.

Thus $\{x_n\}$ is montonically ~~decreasing~~ increasing sequence & bounded above by α.
Hence its limit exist & $\lim_{n\to\infty} x_n = \frac{1+\sqrt{4a+1}}{2} = \alpha$.

By Muhammad Hussain
Assistant Professor Asghar Mall College

Q:18 (R.G.B)

Let $a > 0$ & let $a_1 > 0$. Define

$$a_{n+1} = \sqrt{a + a_n} \quad \forall n \in N$$

Show that $\{a_n\}$ is cgt and find the limit

Sol #

$$a_{n+1}^2 - a_n^2 = (a + a_n) - (a + a_{n-1})$$

$$= a_n - a_{n-1}$$

$$\Rightarrow a_{n+1} - a_n = \frac{a_n - a_{n-1}}{a_{n+1} + a_n}$$

Since $a_{n+1} + a_n > 0$, it follow that $a_{n+1} - a_n$ and $a_n - a_{n-1}$ have same sign

i.e $a_{n+1} \gtrless a_n$ iff $a_n \gtrless a_{n-1}$

$\Rightarrow \{a_n\}$ increasing or decreasing according as $a_2 > a_1$ or $a_2 < a_1$.

Case I If $a_2 > a_1$, then $\{a_n\}$ increasing by Mathematical induction.

Also

$$a_{n+1}^2 - a_n^2 = a_n + b - a_n^2$$

$$= -(a_n^2 - a_n + b) \longrightarrow (1)$$

$\because$

$$a_{n+1} > a_n \quad \forall n$$

$$\sqrt{a + a_n} > a_n \quad \forall n$$

$\Rightarrow \quad a + a_n \geq a_n^2$

$0 > a_n^2 - a_n - a$

or $\quad a_n^2 - a_n - a < 0 \longrightarrow$ (2)

$\because \quad a_n > 0 \quad \forall n \geq 1$

$\therefore \quad a_n$ is the +ve root of equation.

$$a_n^2 - a_n - a = 0$$

Let +ve root be α, then $\alpha = \frac{1 + \sqrt{4a+1}}{2}$

Then other root = $\frac{\text{product of roots}}{\alpha}$

$= -\frac{a}{\alpha}$

$a_n - a_n - a = (a_n - \alpha)(a_n + \frac{a}{\alpha})$

from (2)

$a_n^2 - a_n - a < 0$

$\Rightarrow \quad (a_n - \alpha)(a_n + \frac{a}{\alpha}) < 0$

$\because \quad a_n + \frac{a}{\alpha} > 0$

$\therefore \quad a_n - \alpha < 0$

$\Rightarrow \quad a_n < \alpha \qquad \forall n$

$\Rightarrow$ $\{a_n\}$ is bounded above by +ve root of $x^2 - x - a = 0$

Also $0 < a_1 < \alpha$. Then $\{a_n\}$ is increasing

Thus $\{a_n\}$ is increasing when a_1 is less than +ve root of $x^2 - x - a = 0$ i.e. less than $\alpha = \frac{1 + \sqrt{4a+1}}{2}$

Thus when $a_1 < \alpha$

Then $\{a_n\}$ bounded and increasing & hence is cgt

Case II If $a_2 < a_1$, then $\{a_n\}$ is decreasing by Mathematical induction. i.e.

$$a_{n+1} < a_n$$

$$\Rightarrow \sqrt{a + a_n} < a_n$$

$$\Rightarrow a + a_n < a_n^2$$

$$\Rightarrow a_n^2 - a_n - a > 0$$

$$\Rightarrow (a_n - \alpha)\left(a_n + \frac{a}{\alpha}\right) > 0$$

$\because \quad a_n + \frac{a}{\alpha} > 0$

$\therefore \quad a_n - \alpha > 0$

$$\Rightarrow a_n > \alpha \quad \forall n.$$

$\Rightarrow \{a_n\}$ is bounded below by $\alpha = \frac{1+\sqrt{4a+1}}{2}$

Also $\quad a_1 > \alpha$

Thus if $a_1 > \alpha$, then $\{a_n\}$ is decreasing and bounded and hence cgt.

Let $\lim_{n\to\infty} a_n = l$.

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = \sqrt{\lim_{n\to\infty} a_n + a}$$

$$l = \sqrt{l + a}.$$

$\Rightarrow \ell^2 = \ell + a$

$\Rightarrow \ell^2 - \ell - a = 0$

$\Rightarrow \ell = \dfrac{1 \pm \sqrt{4a+1}}{2}$

$\because a_n > 0 \ \forall n \quad \therefore \lim\limits_{n\to\infty} a_n \geq 0$

$\Rightarrow \ell = \dfrac{1+\sqrt{4a+1}}{2} = \alpha$ the root of equation $x^2 - x - a = 0$

Q: 19 (Gaskil exercise)

Define a sequence by

$a_1 = k, (k>0), \quad a_{n+1} = \sqrt{k + a_n} \quad n \geq 1$

Show that $\{a_n\}$ has a limit and find it.

Sol # $a_n > 0 \ \forall n.$

$a_2 = \sqrt{a_1 + k} = \sqrt{k+k} = \sqrt{2k}$

$a_{n+1}^2 - a_n^2 = (a_n + k) - (a_{n-1} + k)$

$= a_n - a_{n-1}$

$\Rightarrow a_{n+1} - a_n = \dfrac{a_n - a_{n-1}}{a_{n+1} + a_n}$

$\because a_{n+1} + a_n > 0$

$\therefore a_{n+1} - a_n$ & $a_n - a_{n-1}$ have same sign

i.e $a_{n+1} \gtrless a_n$ iff $a_n \gtrless a_{n-1}$

$\Rightarrow \{a_n\}$ increasing or decreasing according as

$a_2 > a_1$ or $a_2 < a_1$

<u>Case I</u> If $a_2 > a_1$, then by mathematical induction it can be proved that $\{a_n\}$ is increasing i.e.

$$a_{n+1} > a_n$$

$$\Rightarrow \sqrt{k + a_n} > a_n$$

$$\Rightarrow k + a_n > a_n^2 \longrightarrow (1)$$

$$\Rightarrow a_n^2 - a_n - k < 0$$

$\because a_n > 0 \quad \forall n$ & $a_n^2 - a_n - k = 0$ has one root +ve & other -ve.

$\therefore a_n$ is +ve root of

$$a_n^2 - a_n - k = 0$$

Let the root be $\alpha = \dfrac{1 + \sqrt{4k+1}}{2}$.

other root $= -\dfrac{k}{\alpha}$

$$\Rightarrow a_n^2 - a_n - k = (a_n - \alpha)\left(a_n + \frac{k}{\alpha}\right)$$

(1) $\Rightarrow$

$(a_n - \alpha)\left(a_n + \frac{k}{\alpha}\right) < 0$

$\because$ $a_n + \frac{k}{\alpha} > 0$

$\Rightarrow$ $a_n - \alpha < 0 \Rightarrow a_n < \alpha \;\; \forall n.$

Also $a_1 < \alpha$

$\Rightarrow$ $\{a_n\}$ is increasing if $a_1 < \alpha$ &

$a_n < \alpha \;\; \forall n.$ ie bounded above by

α, +ve root of $x^2 - x - k = 0$

$\Rightarrow$ $\{a_n\}$ is monotonic bounded.

Case II If $a_2 < a_1$, then by mathematical induction $\{a_n\}$ is decreasing ie.

$a_{n+1} < a_n.$

$\Rightarrow$ $\sqrt{k + a_n} < a_n.$

$\Rightarrow$ $k + a_n < a_n^2$

$\Rightarrow$ $a_n^2 - a_n - k > 0$

$\Rightarrow$ $(a_n - \alpha)\left(a_n + \frac{k}{\alpha}\right) > 0$

$\because$ $a_n + \frac{k}{\alpha} > 0$

$\therefore$ $a_n - \alpha > 0 \Rightarrow a_n > \alpha, \; a_1 > \alpha.$

$\Rightarrow$ $\{a_n\}$ is decreasing if $a_1 < \alpha$ & is bounded below by α

Thus $\{a_n\}$ is monotonic bounded sequence and hence cgt.

$\because \quad a_n > 0 \quad \forall n.$

$\therefore \lim_{n\to\infty} a_n \geq 0$

Let $\lim_{n\to\infty} a_n = l.$

$a_{n+1} = \sqrt{k + a_n}$

$\Rightarrow a_{n+1}^2 = k + a_n.$

$\Rightarrow \left(\lim_{n\to\infty} a_{n+1}\right)^2 = k + \lim_{n\to\infty} a_n$

$l^2 = k + l.$

$l^2 - l - k = 0$

$\Rightarrow l = \dfrac{1 \pm \sqrt{4k+1}}{2}$

$\because l \geq 0 \quad \therefore l = \dfrac{1+\sqrt{4k+1}}{2} = \alpha,$

the root of equation $x^2 - x - k = 0$

Q # 20 (R.G.B)

Let $x_1 > 1$ and $x_{n+1} = 2 - \dfrac{1}{x_n}$ $\forall n \in N$

Show that $\{x_n\}$ is bounded & monotone. Find the limit

Sol $\quad x_{n+1} = 2 - \frac{1}{x_n}, \quad x_1 > 1$

$x_2 = 2 - \frac{1}{x_1} > 1 \quad \because \frac{1}{x_1} < 1$

Let $x_k > 1$ for some $k \in N$

$\Rightarrow \frac{1}{x_k} < 1$

$\Rightarrow -\frac{1}{x_k} > -1$

$\Rightarrow 2 - \frac{1}{x_k} > 1$

$\Rightarrow x_{k+1} > 1$

Thus $x_k > 1 \Rightarrow x_{k+1} > 1$

Thus by M. induction.

$x_n > 1 \quad \forall n$

$$x_n - x_{n+1} = x_n - \left(2 - \frac{1}{x_n}\right)$$

$$= \frac{x_n^2 - 2x_n + 1}{x_n}$$

$$= \frac{(x_n - 1)^2}{x_n} > 0 \quad \because x_n > 1 \ \forall n$$

$\Rightarrow x_n > x_{n+1}$

$\Rightarrow \{x_n\}$ monotonic bounded and hence Convergent

$\because x_n > 1 \quad \forall n$

$\therefore \lim_{n \to \infty} x_n = l \geq 1$

117

$$\lim_{n\to\infty} x_{n+1} = 2 - \frac{1}{\lim_{n\to\infty} x_n}$$

$$l = 2 - \frac{1}{l}.$$

$$l^2 = 2l - 1$$

$$l^2 - 2l + 1 = 0$$

$$(l-1)^2 = 0 \Rightarrow \boxed{l = 1}$$

Q: 21 Grashil

If $x_1 = 4$ $\quad x_{n+1} = 3 - \frac{2}{x_n}$.

$$x_2 = 3 - \frac{2}{x_1} = 3 - \frac{2}{4}$$

$$= 3 - \frac{1}{2} = \frac{5}{2} = 2.5 > 2.$$

$$x_3 = 3 - \frac{2}{x_2} = 3 - \frac{4}{5} = \frac{11}{5} = 2.2 > 2$$

Let $x_k > 2$ for some $k \in N, k > 2$.

$$+\frac{1}{x_k} < \frac{1}{2}$$

$$\Rightarrow \frac{2}{x_k} < 1$$

$$\Rightarrow -\frac{2}{x_k} > -1$$

$$\Rightarrow 3 - \frac{2}{x_k} > 2$$

$$\Rightarrow x_{k+1} > 2$$

Thus $x_k > 2$

$\Rightarrow x_{k+1} > 2$

$\Rightarrow x_n > 2 \quad \forall n \geq 1.$

$$x_n - x_{n+1} = x_n - \left(3 - \frac{2}{x_n}\right)$$

$$= \frac{x_n^2 - 3x_n + 2}{x_n}$$

$$= \frac{(x_n - 2)(x_n - 1)}{x_n} > 0 \qquad \begin{matrix} x_n - 1 > 0 \\ x_n > 0 \end{matrix}$$

$x_n - 2 > 0$ $\therefore$

$\because x_n > 2$

$\Rightarrow x_n > x_{n+1}$

$\Rightarrow \{x_n\}$ is decreasing & bounded and hence Convergent $\therefore \lim_{n\to\infty} a_n \geq 2.$

Let $\lim_{n\to\infty} a_n = l.$

$$x_{n+1} = 3 - \frac{2}{x_n}$$

$$\Rightarrow \lim_{n\to\infty} x_{n+1} = 3 - \frac{2}{\lim_{n\to\infty} x_n}$$

$$l = 3 - \frac{2}{l}$$

$$l^2 = 3l - 2$$

$$l^2 - 3l - 2 = 0$$

$$(l-2)(l-1) = 0$$

$\Rightarrow l = 2,$ or $l = 1$

But $l \geq 2 \quad \Rightarrow l = 2$

Q:22 (Generalisation of $\underline{119}$ Q20 & Q21 made myself.)

Define a sequence as

$a \geq 2$, $x_1 > a-1$, $x_{n+1} = a - \dfrac{a-1}{x_n}$

Prove that $\{x_n\}$ converges to a limit. Find the limit

Sol

$$x_1 > a-1$$

$$x_2 = a - \frac{a-1}{x_1} \longrightarrow ①$$

$$\because \; x_1 > a-1$$

$$1 > \frac{a-1}{x_1}$$

$$\frac{a-1}{x_1} < 1$$

$$-\frac{a-1}{x_1} > -1$$

$$\Rightarrow a - \frac{a-1}{x_1} > a-1$$

$$\Rightarrow x_2 > a-1$$

Let $x_k > a-1$ for some $k \geq 2$.

$$\Rightarrow \frac{1}{x_k} < \frac{1}{a-1}$$

$$\Rightarrow \frac{a-1}{x_k} < 1$$

$$\Rightarrow -\frac{a-1}{x_k} > -1$$

$\Rightarrow \quad a - \dfrac{a-1}{x_k} > a-1$

$\Rightarrow \quad x_{k+1} > a-1$

Hence $x_k > a-1 \Rightarrow x_{k+1} > a-1$

Thus $x_n > a-1 \quad \forall n.$

$$x_n - x_{n+1} = x_n - \left(a - \frac{a-1}{x_n}\right)$$

$$= \frac{x_n^2 - a x_n + (a-1)}{x_n}$$

$$= \frac{x_n^2 - (a-1)x_n - x_n + (a-1)}{x_n}$$

$$\Rightarrow \frac{x_n\left(x_n - (a-1)\right) - 1\left(x_n - (a-1)\right)}{x_n}$$

$$\Rightarrow \frac{(x_n - 1)\left(x_n - (a-1)\right)}{x_n} \geq 0$$

$\because x_n > a-1 \geq 1 \quad \because a \geq 2$

$\Rightarrow x_n \geq x_{n+1}$

$\Rightarrow \{x_n\}$ is monotone decreasing and bounded & hence Convergent

Also $x_n > a-1$

$\Rightarrow \lim\limits_{n \to \infty} x_n \geq a-1$

$L \geq 1 \quad , \, , \, .$

Let $\lim_{n\to\infty} x_n = l$

$\Rightarrow$

Now $x_{n+1} = a - \frac{a-1}{x_n}$

$$\lim_{n\to\infty} x_{n+1} = a - \frac{a-1}{\lim_{n\to\infty} x_n}$$

$$l = a - \frac{a-1}{l}$$

$$l^2 = la - (a-1)$$

$$l^2 - la + a - 1 = 0$$

$$l^2 - (a-1)l - l + a - 1 = 0$$

$$l[l - (a-1)] - 1[l - (a-1)] = 0$$

$$\Rightarrow [l-(a-1)][l-1] = 0$$

$\Rightarrow l - (a-1) = 0 \quad | \quad l - 1 = 0$

$l = a-1 \quad | \quad l = 1$

But $l \geq a-1 \Rightarrow l = a-1$.

Q#23 (R.G.B exercise 3.3)

Let $x_1 \geq 2$ & $x_{n+1} = 1 + \sqrt{x_n - 1}$. $\forall n \in N$

Show that $\{x_n\}$ is dereasing and bounded below by 2. Find the limit.

Sol

$x_1 \geq 2$

$x_2 = 1 + \sqrt{x_1 - 1} \geq 2 \quad \because x_1 \geq 2.$

Let $x_k \geq 2$ for some $k \in N$

$\Rightarrow x_k - 1 \geq 1$

$\Rightarrow \sqrt{x_k - 1} \geq 1$

$\Rightarrow 1 + \sqrt{x_k - 1} \geq 2$

$\Rightarrow x_{k+1} \geq 2.$

Thus $x_n \geq 2 \quad \forall n.$

$$x_n - x_{n+1} = x_n - 1 - \sqrt{x_n - 1}$$

$$= \left(\sqrt{x_n - 1}\right)^2 - \sqrt{x_n - 1}$$

$$= \sqrt{x_n - 1}\left(\sqrt{x_n - 1} - 1\right) \geq 0 \quad \because x_n \geq 2$$

$\Rightarrow x_n \geq x_{n+1}$

$\Rightarrow \{x_n\}$ is monotone decreasing & bounded.

and hence Convergent.

Also $x_n \geq 2$

$\Rightarrow \lim_{n \to \infty} x_n \geq 2.$

Let $\lim_{n \to \infty} x_n = l.$

$$x_{n+1} = 1 + \sqrt{x_n - 1}$$

$$\lim_{n \to \infty} x_{n+1} = 1 + \sqrt{\lim_{n \to \infty} x_n - 1}$$

$$l = 1 + \sqrt{l - 1}$$

$\Rightarrow \ l-1 = \sqrt{l-1}$

$(l-1)^2 = l-1$

$\Rightarrow (l-1)^2 - (l-1) = 0$

$\Rightarrow (l-1)[l-1-1] = 0$

$(l-1)(l-2) = 0$

$\Rightarrow l = 1 \qquad l = 2$

Thus $l = 2 \qquad \because l \geq 2$

$\Rightarrow \lim_{n\to\infty} a_n = 2.$

Q#24 (Generalisation of Q23 made by myself.)

Let $x_1 \geq a^2+1$, $a \geq 1$ & $x_{n+1} = a^2 + \sqrt{x_n - a^2}$ $\forall n \in N$. Show that $\{x_n\}$ decreasing and bounded below a^2+1. Find the limit.

Sol $\qquad x_1 \geq a^2+1.$

$x_2 = a^2 + \sqrt{x_1 - a^2} \geq a^2+1 \quad \because x_1 \geq a^2+1$

Let $x_k \geq a^2+1$ For som $k \in N$.

$\Rightarrow x_k - a^2 \geq 1 \quad \Rightarrow \sqrt{x_k - a^2} \geq 1$

$\Rightarrow a^2 + \sqrt{x_k - a^2} \geq a^2+1.$

$\Rightarrow x_{k+1} \geq a^2+1$

Thus $\quad x_n \geq a^2+1$

$$x_n - x_{n+1} = x_n - (a^2 + \sqrt{x_n - a^2})$$

$$= x_n - a^2 + \sqrt{x_n - a^2}$$

$$= \sqrt{x_n - a^2}\left(\sqrt{x_n - a^2} - 1\right) \geq 0$$

$$\because \; x_n \geq a^2+1$$

$\Rightarrow x_n \geq x_{n+1}$

$\Rightarrow \{x_n\}$ is monotone decreasing and bounded.

and hence convergent

$$\because \; x_n \geq a^2+1.$$

$$\therefore \lim_{n\to\infty} x_n \geq a^2+1$$

Let $\lim_{n\to\infty} x_n = l.$

$$x_{n+1} = a^2 + \sqrt{x_n - a^2}$$

$$\lim_{n\to\infty} x_{n+1} = a^2 + \sqrt{\lim_{n\to\infty} x_n - a^2}$$

$$l = a^2 + \sqrt{l - a^2}$$

$$l - a^2 = \sqrt{l - a^2}$$

$$(l-a^2)^2 = l - a^2.$$

$$(l-a^2)^2 - (l-a^2) = 0$$

$$(l-a^2)\left[l - a^2 - 1\right] = 0 \Rightarrow l = a^2$$

or $l = a^2+1.$

Thus $\lim_{n\to\infty} x_n = a^2+1$

125

Q:25. A sequence $\{s_n\}$ is defined as

$$s_1 = a > 0 \qquad s_{n+1} = \sqrt{\frac{ab^2 + s_n^2}{a+1}}$$

$$b > a \qquad n \geq 1$$

Show that $\{s_n\}$ is cgt and find its limit

Sol It is given that

$$a < b \Rightarrow s_1 < b \longrightarrow (1)$$

$$s_{n+1}^2 - b^2 = \frac{ab^2 + s_n^2}{a+1} - b^2$$

$$= \frac{s_n^2 - b^2}{a+1} < 0 \quad \text{whenever } s_n^2 - b^2 < 0$$

$$\Rightarrow s_{n+1}^2 - b^2 < 0 \quad \text{when } s_n^2 - b^2 < 0$$

$$\Rightarrow s_{n+1} < b \quad \text{when } s_n < b \longrightarrow (2)$$

By (1) & (2)

By mathematical induction, we have

$$s_n < b \quad \forall n$$

OR

$$s_1 < b \Rightarrow s_n < b \quad \text{for } n = 1$$

Let $s_k < b$ for $k > 1$

$$\Rightarrow s_k^2 - b^2 < 0$$

$$\Rightarrow s_k^2 < b^2$$

$\Rightarrow ab^2 + s_k^2 < b^2 + ab^2$

$\Rightarrow \dfrac{ab^2 + s_k^2}{a+1} < \dfrac{b^2(a+1)}{a+1} \qquad \because a+1 > 0$

$\Rightarrow \sqrt{\dfrac{ab^2 + s_k^2}{a+1}} < b$

$\Rightarrow s_{k+1} < b$

Thus by mathematical induction

$s_n < b \quad \forall n$

$\Rightarrow \{s_n\}$ is bounded above.

Again

$s_{n+1}^2 - s_n^2 = \dfrac{ab^2 + s_n^2}{a+1} - s_n^2$

$= \dfrac{a(b^2 - s_n^2)}{a+1} > 0 \quad [\because s_n < b \ \forall n]$

$\Rightarrow s_{n+1}^2 > s_n^2$

$\Rightarrow |s_{n+1}|^2 > |s_n|^2$

$\Rightarrow |s_{n+1}| > |s_n|$

$\Rightarrow s_{n+1} > s_n \quad [\because s_n > 0]$

$\Rightarrow \{s_n\}$ is increasing

Thus $\{s_n\}$ is monotonically increasing and bounded above

note $|a| > |b| \not\Rightarrow a > b$
but if $a > 0,\ b > 0$, then
$|a| > |b| \Rightarrow a > b$
Also $|a| < |b| \not\Rightarrow a < b$
If $a > 0, b > 0$, then
$|a| < |b| \Rightarrow a < b$

and hence convergent.

Let $\lim_{n\to\infty} s_n = l$

Now $s_{n+1}^2 = \dfrac{ab^2 + s_n^2}{a+1}$

$\Rightarrow \left(\lim_{n\to\infty} s_{n+1}\right)^2 = \dfrac{ab^2 + \left(\lim_{n\to\infty} s_n\right)^2}{a+1}$

$l^2 = \dfrac{ab^2 + l^2}{a+1}$

$al^2 + l^2 = ab^2 + l^2$

$\Rightarrow l^2 = b^2 \Rightarrow l = \pm b$

$\because s_n > 0 \quad \forall n.$

$\therefore \lim_{n\to\infty} s_n \geq 0 \quad \forall n$

Hence sequence $\{s_n\}$ converges to b

Q#26 If a_1, b_1 are two +ve unequal numbers and a_n, b_n are defined as

$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}) \quad b_n = \sqrt{a_{n-1} b_{n-1}} \quad n \geq 2.$

prove that two sequences $\{a_n\}$ and $\{b_n\}$ are monotonic, one increasing and the other decreasing and that they tend to the same limit.

Sol Let $a_1 > b_1$.

Since for any two +ve numbers, the A.M is greater than the G.M.

$\therefore \quad a_n > b_n$

Also $\quad a_{n+1} = \frac{1}{2}(a_n + b_n) < \frac{1}{2}(a_n + a_n) = a_n \quad \because b_n < a_n$

$\Rightarrow \{a_n\}$ is $\downarrow$.

$\Rightarrow a_1 > a_2 > a_3 > a_4 > a_5 > \cdots$

Again $\quad b_{n+1} = \sqrt{a_n \cdot b_n} > \sqrt{b_n \cdot b_n} = b_n \quad [a_n > b_n]$

$\therefore \{b_n\}$ is monotone decreasing

Now $a_n = \frac{1}{2}(a_{n-1} + b_{n-1}) > \frac{1}{2}(b_{n-1} + b_{n-1}) = b_{n-1} \quad \because a_n > b_n \ \forall n$

$\Rightarrow a_n > b_{n-1} > b_{n-2} \cdots > b_2 > b_1$

$\Rightarrow a_n > b_1 \quad \forall n$

$\Rightarrow \{a_n\}$ is $\downarrow$ and bounded & hence cgt.

Again $b_n = \sqrt{a_{n-1} \cdot b_{n-1}} < \sqrt{a_{n-1} \cdot a_{n-1}} = a_{n-1}$ by (i)

i.e $\quad b_n < a_{n-1} < a_{n-2} < \cdots < a_2 < a_1$.

$\therefore \{b_n\}$ is bounded above and being monotone increasing is convergent

let $\quad \lim_{n\to\infty} a_n = l_1 \quad \& \quad \lim_{n\to\infty} b_n = l_2$

Since $\quad a_n = \frac{1}{2}(a_{n-1} + b_{n-1})$

$2a_n = a_{n-1} + b_{n-1}$

$\therefore \quad 2a_{n+1} = a_n + b_n$

$\Rightarrow \lim_{n\to\infty} 2a_{n+1} = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

$2l_1 = l_1 + l_2$

$\Rightarrow l_1 = l_2$

$\Rightarrow \{a_n\}$ & $\{b_n\}$ converge to the same limit.

Q#27 If $a_1 > 0, b_1 > 0$ and $a_n = \sqrt{a_{n-1} \cdot b_{n-1}}$ and $b_n = \dfrac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}}$, prove that

(i) $\{a_n\}$ and $\{b_n\}$ are monotonic, the one increasing and the other decreasing (ii) $\{a_n\}$ and $\{b_n\}$ both converge to the same limit.

Sol Suppose $a_1 > b_1$. $\longrightarrow (1)$

$\because$ for any two +ve nos G.M > H.M.

$\therefore \quad a_n > b_n \quad \forall n \longrightarrow (2)$

Also $a_{n+1} = \sqrt{a_n \cdot b_n} < \sqrt{a_n \cdot a_n} = a_n$

$\Rightarrow \{a_n\}$ is $\downarrow$. ie

$a_1 > a_2 > a_3 > a_4 \cdots > a_n >$

$\Rightarrow a_1 \geq a_n \quad \forall n. \longrightarrow (3)$

Again $b_{n+1} = \dfrac{2a_nb_n}{a_n+b_n} > \dfrac{2b_n \cdot b_n}{b_n+b_n} = b_n \quad [a_n > b_n]$

$b_{n+1} > b_n$

$\{b_n\}$ is monotone increasing $b_1 < b_2 < b_3 \ldots < b_n$

or $\quad b_n \geq b_1 \quad \forall n \longrightarrow (4)$

is greater than ...

From ①, ②, ③ & ④ we get

$a_1 \geq a_n > b_n \geq b_1 \quad \forall n \longrightarrow$ ⑤

Now since $a_n > b_1 \quad \forall n$

$\Rightarrow \{a_n\}$ is bounded below and being $\downarrow$ converges to a limit.

Again ⑤ $\Rightarrow b_n < a_1 \quad \forall n$

$\Rightarrow \{b_n\}$ is bounded above and being $\uparrow$ is convergent.

Let $\lim_{n\to\infty} a_n = a$ & $\lim_{n\to\infty} b_n = b$

Now $a_{n+1} = \sqrt{a_n \cdot b_n}$

or $a_{n+1}^2 = a_n b_n$

$\therefore$ we have $\lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

$a^2 = ab$

$\Rightarrow a = b$ which proves the result.

Q 28 Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by $x_1 = \frac{1}{2}$ & $y_1 = 1$

and $x_n = \sqrt{x_{n-1}\, y_{n-1}} \quad n = 2, 3 \ldots$

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-2}}\right) \quad n = 2, 3, 4 \ldots$$

prove that $x_{n-1} < x_n < y_n < y_{n-1} \quad n = 2, 3 \ldots$ and deduce that both sequences converge to the same limit l where $\frac{1}{2} < l < 1$

Sol If $0 < a < b$, then $G.M = \sqrt{ab}$

and $H = \left[\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)\right]^{-1} = \frac{2ab}{a+b}$

Also $a < H < G < b$

we are given that

$\frac{1}{2} = x_1 < y_1 = 1.$

$\Rightarrow$ let $x_{n-1} < y_{n-1}$

$x_{n-1} < x_n < y_{n-1}$ $\left[\because \; x_n = \sqrt{x_{n-1}\, y_{n-1}}\right]$

Further

$x_n < y_n < y_{n-1}$

because y_n is H.M of x_n & y_{n-1}.

$\Rightarrow$

$x_{n-1} < x_n < y_n < y_{n-1}$ $n = 2, 3, \ldots$

$\Rightarrow \{x_n\} \uparrow$ and is bounded above by $y_1 = 1$

The sequence $\{y_n\}$ decreases and is bounded below by $x_1 = \frac{1}{2}$. Hence both the sequence converge.

Suppose $x_n \to l$ as $n \to \infty$ and $y_n \to m$ as $n \to \infty$

Then $l^2 = lm \Rightarrow \boxed{l = m.}$

$\frac{1}{m} = \frac{1}{2}\left(\frac{l+m}{lm}\right) \Rightarrow \boxed{l = m}$

Q29 If a sequence $\{a_n\}$ is defined by

$a_{n+1} = 1 + \frac{1}{a_n} \;\; \forall n \geq 2$ $a_1 > 0, a_1 = 1$

$a_2 = 2$

prove that the sequence $\{a_n\}$ is cgt and.

$\lim_{n \to \infty} a_n = \frac{1 + \sqrt{5}}{2}$

Sol $a_3 < 1 + \frac{1}{2} < 2.$ $a_3 = 3/2$

$a_4 = 1 + \frac{1}{a_3} = 1 + 2/3 < 2.$

Let $a_{n+1} < 2$ for $n \geq 2$

is greater tha...

132

Let $a_{k+1} < 2$ for $k > 2$.

$\Rightarrow \dfrac{1}{a_{k+1}} < \dfrac{1}{2}$

$\Rightarrow 1 + \dfrac{1}{a_{k+1}} < 1 + \dfrac{1}{2} < 2.$

$\Rightarrow a_{k+1+1} < 2$

Thus $a_{n+1} < 2 \quad \forall n \geq 2$

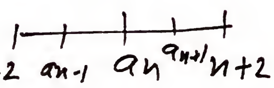
Thus $1 \leq a_n \leq 2 \quad \forall n.$

$$a_n = 1 + \frac{1}{a_{n-1}} = 1 + \frac{1}{1 + \frac{1}{a_{n-2}}}$$

$$= 1 + \frac{a_{n-2}}{1 + a_{n-2}}$$

$$a_{n+2} - a_n = \left(1 + \frac{a_n}{1 + a_n}\right) - \left(1 + \frac{a_{n-2}}{1 + a_{n-2}}\right)$$

$$= \frac{a_n - a_{n-2}}{(1 + a_n)(1 + a_{n-2})}$$



$\Rightarrow a_{n+2} - a_n$ have same sine as $a_n - a_{n-2}$.

Now $a_3 - a_1 = 3/2 - 1 = > 0$

So $a_{2k+1} - a_{2k-1} > 0 \quad \forall k \in N$

It follows that $\{a_{2k-1}\}$ is monotone increasing sequence.

Similarly ~~a_4~~ $a_4 - a_2 = 5/3 - 2 < 0$

and so $a_{2k+2} - a_{2k} < 0 \quad \forall k \in N$

$\Rightarrow \{a_{2k}\}$ is monotone decreasing sequence

Also

$$1 \le a_{2k-1} \le 2 \quad \forall k \in N$$

$$1 \le a_{2k} \le 2 \quad \forall k \in N$$

$\Rightarrow \{a_{2k}\}$ and $\{a_{2k-1}\}$ are bounded monotone sequence and hence cgt.

Let $\quad l_1 = \lim_{k \to \infty} a_{2k-1}$

$\quad l_2 = \lim_{k \to \infty} a_{2k}$.

Now

$$a_n = 1 + \frac{a_{n-2}}{1+a_{n-2}} \quad \text{for } n \geq 3$$

$$a_{2k-1} = 1 + \frac{a_{2k-3}}{1+a_{2k-3}}$$

$$a_{2k} = 1 + \frac{a_{2k-2}}{1+a_{2k-2}}$$

$$\lim_{k \to \infty} a_{2k-1} = 1 + \frac{\lim_{k \to \infty} a_{2k-3}}{1+\lim_{k \to \infty} a_{2k-3}}$$

$$l_1 = 1 + \frac{l_1}{1+l_1}$$

$$\lim_{k \to \infty} a_{2k} = 1 + \frac{\lim_{k \to \infty} a_{2k-2}}{1+\lim_{k \to \infty} a_{2k-2}}$$

$$l_2 = 1 + \frac{l_2}{1+l_2}.$$

Thus l_1 & l_2 both satisfy equation $l^2 - l - 1 = 0$

$$l = \frac{1 \pm \sqrt{5}}{2}$$

But $\quad 1 \leq a_n \leq 2 \quad \forall n.$

So $l_1, l_2 > 0$

$\Rightarrow \quad l_1 = \frac{1 \pm \sqrt{5}}{2} = l_2.$

Q #30 (Grashil) If a sequence $\{a_n\}$ is defined by

$$a_2 > a_1 > 0 \,, \quad a_{n+1} = \frac{a_n + a_{n-1}}{2} \quad \forall n \geq 2$$

$$\text{or } a_n = \frac{a_{n-1} + a_{n-2}}{2} \quad \forall n > 2.$$

prove that the sequence converges also find. limit

Sol # $\quad a_{n+2} - a_n = \frac{1}{2}\left[a_{n+1} + a_n\right] - a_n$

$$= \frac{1}{2}\left[a_n - a_{n-2}\right] \longrightarrow (1)$$

$\because \quad a_1 < a_2$

putting $n = 3, 4, 5 \cdots$ in relation

$$a_n = \frac{1}{2}\left[a_{n-1} + a_{n-2}\right]$$

$$a_3 = \frac{1}{2}\left[a_2 + a_1\right]$$

$$a_4 = \frac{1}{2}\left\{a_3 + a_2\right]$$

$$a_5 = \frac{1}{4}\left[a_4 + a_2\right]$$

$a_m = \frac{1}{2}\left[a_{m-1} + a_{m-2}\right]$

$\because a_1 < a_2$

$a_3 = \frac{1}{2}\left[a_1 + a_2\right] < \frac{1}{2}\left[a_2 + a_2\right] = a_2$

But $a_3 = \frac{1}{2}\left[a_1 + a_2\right]$

$\Rightarrow a_1 < a_3 < a_2$

$a_3 < a_4 < a_2$

$a_3 < a_5 < a_4$

$a_5 < a_6 < a_4$

Thus it appears.

$a_1 < a_3 < a_5 \ldots\ldots$

$a_2 > a_4 > a_4 \ldots\ldots$

from ① let $n = 2m$.

$a_{2m+2} - a_{2m} = \frac{1}{2}\left[a_{2m+1} - a_{2m}\right]$

$\therefore a_{2m+2} < a_{2m} \Rightarrow a_{2m+2} - a_{2m} \not< 0$

$\Rightarrow a_{2m+1} - a_{2m} < 0$

but $\Rightarrow a_{2m+1} < a_{2m}$

$\Rightarrow a_{2m} < a_{2m-2} < \ldots\ldots a_4 < a_2.$

$\therefore$ every odd term is less than every even term

i.e. $\{a_{2m+1}\}$ is increasing and bounded above by a_2 and is therefore convergent

Similarly even term subsequence $\{a_{2n}\}$ is convergent. We show that both converge to same limit

Let $\lim\limits_{n\to\infty} a_{2n+1} = l_1$ & $\lim\limits_{n\to\infty} a_{2n} = l_2$

from recursion relation

$$a_{2n} = \frac{1}{2}[a_{2n-1} + a_{2n-2}]$$

$$\lim_{n\to\infty} a_{2n} = \frac{1}{2}[\lim_{n\to\infty} a_{2n-1} + \lim a_{2n-2}]$$

$$l_2 = \frac{1}{2}[l_1 + l_2]$$

$$2l_2 = l_1 + l_2.$$

$$l_1 = l_2$$

Thus $\{a_n\}$ is cgt.

Now from $a_n = \frac{1}{2}[a_{n-1} + a_{n-2}]$

$$a_{n+1} = \frac{1}{2}[a_{n} + a_{n-1}]$$

$$a_3 = \frac{1}{2}[a_2 + a_1]$$

$$a_4 = \frac{1}{2}[a_3 + a_2]$$

$$a_5 = \frac{1}{2}[a_4 + a_3]$$

$$- \quad - \quad - \quad -$$

$$a_k = \frac{1}{2}[a_{k-1} + a_{k-2}]$$

$$a_{k+1} = \frac{1}{2}[a_k + a_{k-1}]$$

Adding all these

$$a_3 + a_4 + a_5 + \cdots + a_k + a_{k+1} = \frac{1}{2}\Big[a_2 + a_1 + a_3 + a_2 + a_4 + a_3 + \cdots - - a_{k-1} + a_{k-2} + a_k + a_{k-1}\Big]$$

$$= \frac{1}{2}\Big[a_1 + 2a_2 + 2a_3 + 2a_4 + \cdots - - - 2a_{k-1} + a_k\Big]$$

$$\tfrac{1}{2} a_k + a_{k+1} = \frac{1}{2}\left[a_1 + 2a_2\right]$$

$\Rightarrow$ Taking limit when $k \to \infty$

$$\tfrac{1}{2} l + l = \frac{1}{2}\left[a_1 + 2a_2\right]$$

$$\tfrac{3}{2} l = \frac{1}{2}\left[a_1 + 2a_2\right]$$

$$l = \frac{1}{3}\left[a_1 + 2a_2\right]$$

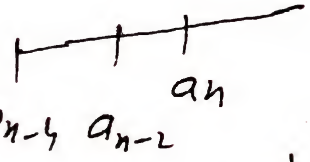
$$= a\left[\frac{1+a_{n-3} - 1 - a_{n-1}}{(1+a_{n-1})\,(1+a_{n-3})}\right]$$

$$= \frac{-a\,(a_{n-1} - a_{n-3})}{(1+a_{n-1})\,(1+a_{n-3})} \longrightarrow (2)$$

$$= \frac{-a\left[\frac{a}{1+a_{n-2}} - \frac{a}{1+a_{n-4}}\right]}{(1+a_{n-1})\,(1+a_{n-3})}$$

$$= \frac{a^2\,(a_{n-2} - a_{n-4})}{(1+a_{n-1})\,(1+a_{n-3})\,(1+a_{n-2})(1+a_{n-4})} \longrightarrow (3)$$

(3) $\Rightarrow$ that $a_n - a_{n-2}$ & $a_{n-2} - a_{n-4}$ have same signs. So even and odd terms form seperate monotone sub sequences



from (2) we note that if odd numberd sub-sequence form a monotone decreasing sub-sequence, then sequence of even no terms from a sub-sequence of increasing terms and vice versa.

Since every term of the sequence is +ve.

$$a - a_n = a - \frac{a}{1+a_{n-1}} = \frac{a\,a_{n-1}}{1+a_{n-1}} > 0$$

$\Rightarrow$ $a - a_n > 0 \quad \forall n$

$\Rightarrow$ $a_n < a \quad \forall n$

Thus $0 < a_n < a$.

OR

$$a_n = \frac{a}{1+a_{n-1}} < a \quad \because a_{n-1} > 0$$

Thus monotone increasing sub-sequence is bounded above by a and the monotone decreasing subsequence is bounded below by 0

Hence the two sub-sequence converge.

Let even no terms converges to l_1 & odd no term converge to l_2.

Then $\lim\limits_{n\to\infty} a_n = \dfrac{a}{1+\lim\limits_{n\to\infty} a_{n-1}}$

we get

(a) For n even $l_1 = \dfrac{a}{1+l_2}$ or $l_1 l_2 + l_1 = a$.

(b) for n odd $l_2 = \dfrac{a}{1+l_1}$ or $l_1 l_2 + l_2 = a$.

$\Rightarrow$ $l_1 l_2 + l_1 = l_1 l_2 + l_2$

$\Rightarrow$ $\boxed{l_1 = l_2}$

Q# 32 140

A sequence $\{a_n\}$ is defined as $a_1 = 1$

$$a_{n+1} = \frac{4+3a_n}{3+2a_n} \quad n \geq 1$$

Show that $\{a_n\}$ Converges and Find Limit

Sol

$$a_1 = 1 \qquad a_2 = \frac{4+3a_1}{3+2a_1} = 7/5 > 1$$

$$a_1 > a_2$$

Let $a_{n+1} > a_n$

Then

$$a_{n+2} - a_{n+1} = \frac{4+3a_{n+1}}{3+2a_{n+1}} - \frac{4+3a_n}{3+2a_n}$$

$$= \frac{a_{n+1} - a_n}{(3+2a_{n+1})(3+2a_n)} > 0$$

$[\because a_{n+1} > a_n$, $a_n > 0 \ \forall n]$

$$a_{n+2} - a_{n+1} > 0$$

$$\Rightarrow a_{n+2} > a_{n+1}$$

By mathematical induction $\{a_n\}$ is increasing

$$a_{n+1} = \frac{4+3a_n}{3+2a_n} = \frac{3}{2} - \frac{1}{2(3+2a_n)}$$

$= \frac{3}{2} - ($a +ve quantity less than 1$)$ $(a_2 > a_1 = 1)$

$$< \frac{3}{2} \qquad \Rightarrow a_{n+1} < 3/2 \quad \forall n.$$

$\Rightarrow \{a_n\}$ is bounded above.

Thus $\{a_n\}$ bounded monotonic and hence convergent. Let $\lim_{n\to\infty} a_n = l$.

Then $a_{n+1} = \dfrac{4+3a_n}{3+2a_n}$

$\Rightarrow l = \dfrac{4+3l}{3+2l}$

$3l + 2l^2 = 4+3l$. $\Rightarrow l^2 = \frac{4}{2} = 2$

$2l^2 = 4$

$\Rightarrow l = \pm\sqrt{2}$. $l = \sqrt{2}$

But l can not be -ve.

Q:33# A sequence is defined by

$a_0 = 1 \quad a_1 = 3/2 \quad \cdots \quad a_{n+1} = 1 + \frac{1}{2}a_n$

Show that the sequence converges by showing that the sequence is bounded and find its limit.

Sol

$a_0 = 1 \qquad a_1 = 3/2 = 1.5$

$a_2 = 7/4 = 1.75$

$a_3 = 1 + \frac{1}{2}a_2 = 1 + \frac{1}{2}\cdot 7/4 = \frac{15}{8} = 1.88$

The sequence appears to increase.

Hence it is possible only if

$a_{n+1} > a_n$

$\Rightarrow 1 + \frac{1}{2}a_n > a_n \qquad \Rightarrow 2 > a_n$

$1 > 1/2\, a_n \qquad a_n < 2 \quad \forall n$

$\Rightarrow$ Sequence is bounded above by 2.
Hence the sequence is bounded and thus convergent.

Let $\lim_{n\to\infty} a_n = l$.

$\Rightarrow \quad l = 1 + \frac{1}{2} l \Rightarrow l = 2$.

Thus $\lim_{n\to\infty} a_n = 2$

Theorem If $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$

Proof **Case I** Let $0 < a < 1$

Then $\{a^n\}$ is a bounded monotone sequence and thus convergent.

Let $\lim_{n\to\infty} a^n = l$.

Then $l = \lim_{n\to\infty} a^{n+1} = \lim_{n\to\infty} a a^{n}$

$l = al$

$\Rightarrow l(1-a) = 0 \Rightarrow l = 0$ or $a = 1$

If $l \neq 0$, then $a = 1$ which is impossible.
because $|a| < 1 \Rightarrow -1 < a < 1$

Therefor $\lim_{n\to\infty} a^n = 0 \quad$ if $0 < a < 1$

Case II Let $-1 < a < 0$

Then $0 < -a < 1$

Now $(-a)^n = (-1)^n a^n$

$\{(-1)^n\}$ is bounded & $\lim_{n\to\infty} a^n = 0$

$\Rightarrow \lim_{n\to\infty} (-1)^n a^n = 0$

$\Rightarrow \lim_{n\to\infty} (-a)^n = 0$

$\lim_{n\to\infty} a^n = \lim_{n\to\infty} [(-1)(-a)]^n$

$= \lim_{n\to\infty} (-1)^n (-a)^n = 0$

Case II If $a = 0$, then $\lim_{n\to\infty} a^n = 0$

OR

If $a = 0$, then $\lim_{n\to\infty} a^n = 0$

Let us as assume that $0 < |a| < 1$

Let $\epsilon > 0$

$|a^n - 0| < \epsilon$

$|a^n| < \epsilon \Rightarrow |a|^n < \epsilon$

$n \ln|a| < \ln \epsilon$

$n > \dfrac{\ln \epsilon}{\ln|a|}$ $\quad \because \ln|a| < 0$ because $0 < |a| < 1$

This inequality provides us a clue to choose M_1 (fixed). Let us Consider

Case I If $\epsilon < 1$

Then $\ln \epsilon < 0$ and let $n_1 > \frac{\ln \epsilon}{\ln|a|}$ such that

$$|a^n - 0| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow \lim_{n\to\infty} a^n = 0$$

<u>Case II</u> If $\epsilon \geq 1$, then $\ln \epsilon \geq 0$

and $\frac{\ln \epsilon}{\ln|a|} \leq 0$

In this case we can choose an n_1 +ve and will have

$$|a^n - 0| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow \lim_{n\to\infty} a^n = 0$$

<u>Theorem</u> If $a > 0$, then $\lim_{n\to\infty} a^{1/n} = 1$

<u>Proof</u> # <u>Case-I</u> let $a \geq 1$

Then $a^{1/n} \geq 1$

$\{a^{1/n}\}$ is bounded below by 1.

Also $a > a^{1/2} > a^{1/3} > a^{1/4} > \cdots$

$\Rightarrow \{a^{1/n}\}$ is bounded and decreasing and hence convergent

Let $\lim_{n\to\infty} a^{1/n} = l$.

$\therefore a^{1/n} \geq 1 \quad \Rightarrow \lim_{n\to\infty} a^{1/n} \geq 1$

$\cancel{145}$ $144+1 = 145$

$\Rightarrow \lim_{n\to\infty} a^{2/n} = \lim_{n\to\infty} (a^{1/n})^2 = l^2$

The sub-sequence $\{a^{2/2n}\}$ & $\{a^{2/n}\}$ converge to same limit l^2 hence.

$$l = l^2$$
$$l(l-1) = 0$$
$$l = 0 \text{ or } l = 1$$

Since $l \neq 0$, $l = 1$

$\Rightarrow \lim_{n\to\infty} a^{1/n} = 1$

<u>OR</u>

Let $a > 1$, then $a^{1/n} > 1 \quad \forall n.$

Let $a^{1/n} = 1 + h_n$ where $h_n > 0$

$a = (1+h_n)^n = 1 + nh_n + \frac{n(n-1)}{2!} h_n^2 e \cdots h_n^2$

$\geq 1 + nh_n$

$\Rightarrow \frac{a-1}{n} \geq h_n$

$\Rightarrow 0 < h_n \leq \frac{a-1}{n} \quad \forall n.$

$\therefore \lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{a-1}{n} = 0$

By squeeze principle

$\lim_{n\to\infty} h_n = 0$

$\Rightarrow \lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} (1+h_n) = 1+0 = 1$

Case

$\Rightarrow \{a\}$

If $a=1$, then $\lim_{n\to\infty} a^{1/n} = 1$

Case II If $0 < a < 1$

Then $\frac{1}{a} > 1$ and by case I above

$$\lim_{n\to\infty} \left(\frac{1}{a}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{a^{1/n}} = 1$$

$$\Rightarrow \lim_{n\to\infty} a^{1/n} = 1.$$

Theorem# Discuss the nature of the sequence $\{a^n\}$ for $a \in R$.

Sol The behaviour of the sequence $\{a^n\}$ depends upon the value of a

Case 1 Let $a > 1$

Then $a = 1+h$ where $h > 0$

$$a^n = (1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \dots + h^n$$

$$a^n \geq 1 + nh$$

as $n \to \infty$ $\quad 1+nh \to \infty$

$\Rightarrow a^n \to \infty$ as $n \to \infty$

$\Rightarrow \{a^n\}$ diverges

Case II Let $a = 1$

$a^n = 1$

$\Rightarrow \{a^n\}$ is a Constant sequence and Converges

Case III Let $0 < a < 1$

Then $1/a > 1$

$1/a = 1+h$ for some $h > 0$

$\frac{1}{a^n} = (1+h)^n > nh$

$\Rightarrow 0 < a^n < \frac{1}{nh}$

By squeeze play

$\lim_{n \to \infty} a^n = 0$

Case IV If $a = 0$

$\lim_{n \to \infty} a^n = 0 \Rightarrow \{a^n\}$ Converges to 0

Case V Let $-1 < a < 0$

put $a = -b$, then

$-1 < a < 0 \Rightarrow -1 < -b < 0$

$\Rightarrow 0 < b < 1$

$b^n \to 0$ as $n \to \infty$

$\lim_{n \to \infty} a^n = \lim_{n \to \infty} (-b)^n = \lim_{n \to \infty} (-1)^n a^n = 0$

$\Rightarrow \{a^n\}$ Converges to 0

Case VI Let $a = -1$

$$a^n = (-1)^n = \begin{cases} -1 & \text{If } n \text{ is odd} \\ 1 & \text{If } n \text{ is even} \end{cases}$$

The sequence is $-1, 1,$

$\Rightarrow \{a^n\}$ oscillates finitely

Case VII $a < -1$ let $a = -b$

~~$a^n = (-1)^n$~~

$a < -1 \Rightarrow -b < -1 \Rightarrow b > 1$

By $b^n \to \infty$ as $n \to \infty$

$$a^n = (-b)^n = \begin{cases} -b^n & \text{if } n \text{ is odd} \\ b^n & \text{if } n \text{ is even} \end{cases}$$

$a^n \to -\infty$ as $n \to \infty$ when n is odd

$a^n \to \infty$ as $n \to \infty$ when n is even.

$\Rightarrow \{a^n\}$ oscillates infinitely

Hence $\{a^n\}$ converges when $-1 < a \leq 1$

Lemma # Let a & b be numbers such that $0 \leq a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)\, b^n$$

OR

$$b^n \cancel{\times} \{ b - (n+1)(b-a) \} < a^{n+1}$$

Proof $0 \leq a < b$

By actual division

$$\frac{b^{n+1} - a^{n+1}}{b - a} = a^0 b^n + a b^{n-1} + a^2 b^{n-2} + \cdots + a^{n-1} b + a^n$$

(n terms)

$$< b^n + b b^{n-1} + b^2 b^{n-2} + b^3 b^{n-3} + \cdots + b^{n-1} b + b^n \quad (\because a < b)$$

$$= (n+1)\, b^n$$

$$\Rightarrow b^{n+1} - a^{n+1} < (n+1) b^n [b - a]$$

$$b^{n+1} - (n+1)b^n(b-a) < a^{n+1}$$

$$b^n[b - (b-a)(n+1)] < a^{n+1}$$

Theorem # prove that the sequence $\{(1+\frac{1}{n})^n\}$ is bounded and increasing

OR

Let $e_n = (1+\frac{1}{n})^n$, prove that $\{e_n\}$ is increasing and bounded.

Proof # We know that for $0 \leq a < b$

$$b^n[b - (n+1)(b-a)] \leq a^{n+1}$$

Taking $a = 1 + \frac{1}{n+1}$ & $b = 1 + \frac{1}{n}$.

$b > a \quad \because \{\frac{1}{n+1} < \frac{1}{n}\}$

$$(1+\tfrac{1}{n})^n\left[(1+\tfrac{1}{n}) - (n+1)\left(\cancel{1}+\tfrac{1}{n} - \cancel{1} - \tfrac{1}{n+1}\right)\right] < \left(1+\tfrac{1}{n+1}\right)^{n+1}$$

$$(1+\tfrac{1}{n})^n\left[(1+\tfrac{1}{n}) - (n+1)\left(\tfrac{1}{n} - \tfrac{1}{n+1}\right)\right] < \left(1+\tfrac{1}{n+1}\right)^{n+1}$$

$$(1+\tfrac{1}{n})^n\left[(1+\tfrac{1}{n}) - (n+1)\left(\frac{\cancel{n}+1-\cancel{n}}{n(n+1)}\right)\right] < \left(1+\tfrac{1}{n+1}\right)^{n+1}$$

$$(1+\tfrac{1}{n})^n\left[1+\cancel{\tfrac{1}{n}} - \cancel{\tfrac{1}{n}}\right] < \left(1+\tfrac{1}{n+1}\right)^{n+1}$$

$$\left(1+\tfrac{1}{n}\right)^n < \left(1+\tfrac{1}{n+1}\right)^{n+1}$$

$$\Rightarrow e_n < e_{n+1} \quad \forall n.$$

149

$\Rightarrow \{e_n\}$ is $\uparrow$

Let $a = 1$ $\qquad b = 1 + \frac{1}{2n}$

$\left(1+\frac{1}{2n}\right)^n \left\{1 + \frac{1}{2n} - (n+1)\left(\cancel{1} + \frac{1}{2n} - \cancel{1}\right)\right\} < 1^{n+1}$

$\left(1+\frac{1}{2n}\right)^n \left[1 + \cancel{\frac{1}{2n}} - \frac{1}{2} - \cancel{\frac{1}{2n}}\right] < 1$

$\left(1+\frac{1}{2n}\right)^n \left[\frac{1}{2}\right] < 1$

2a $\Rightarrow \left(1+\frac{1}{2n}\right)^n < 2.$

This $\Rightarrow \left(1+\frac{1}{2n}\right)^{2n} < 4 \Rightarrow e_{2n} < 4$

not Since $\{e_n\}$ is increasing and $2n > n$.

Adjudent, therefore

Remedize. $e_n < e_{2n} < 4 \quad \forall n.$

$\Rightarrow \{e_n\}$ is bounded.

24 Thus $\{e_n\}$ is Convergent.

If $n = 1$ $\qquad e_1 = (1+1)^1 = 2.$

$\Rightarrow 2 \leq e_n < 4 \quad \forall n$

OR

$\because n$ is +ve integer, by binomial theorem

$e_n = \left(1+\frac{1}{n}\right) = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \cdots \frac{n(n-1)\cdots 2\cdot 1}{n!}\cdot\frac{1}{n^n}$

$$e_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\cdots$$
$$\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

changing n to $n+1$

$$e_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)$$
$$\cdots\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$$
$$+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

note that e_n has $n+1$ terms & e_{n+1} has $n+2$ terms and one more term.

Also $\frac{1}{n+1} \not< \frac{1}{n} \Rightarrow -\frac{1}{n+1} > -\frac{1}{n}$.

$\Rightarrow 1-\frac{1}{n+1} > 1-\frac{1}{n}$

Similarly $1-\frac{k}{n+1} > 1-\frac{k}{n}$ for $k=1,2,\ldots n-1$

$\Rightarrow$ each term in e_n after the 1st two terms is less than the corresponding term in e_{n+1} and e_{n+1} has also one additional term. It comes out

$$e_{n+1} \geq e_n$$

$\Rightarrow \{e_n\}$ is $\uparrow$

To show that $\{e_n\}$ is bounded above we have.

$$= 1+n\cdot\frac{1}{n}+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots$$
$$\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-}{n}\right)$$

$\leq 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{k!}+\cdots\frac{1}{n!}$ $\quad \because 1-P/n < 1$

$1+\frac{1}{2^0}+\frac{1}{2^1}+\cdots+\frac{1}{2^{k-1}}+\cdots+\frac{1}{2^{n-1}}$ $\quad \because \frac{1}{P!} \leq \frac{1}{2^{P-1}}$

$\Rightarrow \quad e_n$

$$= 1 + \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 1+2\left[1-\left(\tfrac{1}{2}\right)^n\right] < 3$$

$\Rightarrow e_n < 3 \quad \forall n.$

for $n=1$ $\quad e_1 = 2.$

for $n > 1$

$$2 < e_n < 3 \quad \forall n > 1$$

Thus $\{e_n\}$ bounded monotone and hence convergent

Note By refining our estimates we can find closer rational approximations to e but we can not evaluate e since e is an irrational number. However it is possible to evaluate e to as many decimal places as desired.

Theorem # Prove that e is irrational

Proof # Let e be a rational number and

$$e = p/q \quad \text{where } p, q \in N \text{ and } q > 1$$

Let $$s_q = \sum_{k=1}^{q} \frac{1}{k!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{q!}$$

Since $$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

So $$0 < e - s_q = \frac{1}{(q+1)!}+\frac{1}{(q+2)!}+\frac{1}{(q+3)!}+\cdots$$

$$= \frac{1}{(q+1)!}\left[1+\frac{1}{q+2}+\frac{1}{(q+2)(q+3)}+\cdots\right]$$

$$< \frac{1}{\ }\left\{1+\frac{1}{q+1}+\frac{1}{(q+1)^2}+\cdots\right\}$$

$$= 1 + \frac{1-(1/2)^n}{1-1/2} = 1+2\left[1-\left(\tfrac{1}{2}\right)^n\right] < 3$$

$\Rightarrow e_n < 3 \quad \forall n.$

for $n=1$ $\quad e_1 = 2.$

for $n > 1$

$$2 < e_n < 3 \quad \forall n > 1$$

Thus $\{e_n\}$ bounded monotone and hence convergent

Note By refining over estimates we can find closer rational approximations to e but we can not evaluate e since e is an irrational number. However it is possible to evaluate e to as many decimal places as desired.

Theorem # Prove that e is irrational

Proof # Let e be a rational number

and $\quad e = p/q \quad$ where $p, q \in N$ and $q > 1$

Let $$S_q = \sum_{k=1}^{q} \frac{1}{k!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{q!}$$

Since $$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

so $$0 < e - S_q = \frac{1}{(q+1)!}+\frac{1}{(q+2)!}+\frac{1}{(q+3)!}+\cdots$$

$$= \frac{1}{(q+1)!}\left[1+\frac{1}{q+2}+\frac{1}{(q+2)(q+3)}+\cdots\right]$$

$$< \frac{1}{(q+1)!}\left[1+\frac{1}{q+1}+\frac{1}{(q+1)^2}+\cdots\right]$$

$$a_n \leq a_m < -k \quad \forall n \geq m.$$

$$\Rightarrow a_n < -k \quad \forall n \geq m.$$

$$\Rightarrow \lim_{n\to\infty} a_n = -\infty \Rightarrow \{a_n\} \text{ diverges to } -\infty$$

$$= \frac{1}{(q+1)!}\left[\frac{1}{1-\frac{1}{q+1}}\right] \qquad S_\infty = \frac{a}{1-r}.$$

$$= \frac{1}{(q+1)!} \times \frac{q+1}{q} = \frac{1}{(q!)q}.$$

$\Rightarrow$

$$0 < (e - s_q) < \frac{1}{q!\,q}.$$

$$\Rightarrow 0 < (e - s_q)q! < \frac{1}{q} < 1 \qquad (\because q > 1,\ \tfrac{1}{q} < 1$$

$$0 < eq! - s_q(q!) < 1.$$

$\because$ $qe = p$ by $p/q = e$.

$\Rightarrow eq! = eq(q-1)! = p(q-1)!$ is an integer

Also

$$(q!)s_q = q!\left[1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right]$$

$$= q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + 1$$ is

an integer for each $q > 1$

$\Rightarrow$ $(e - s_q)q!$ is an integer lying b/w 0 and 1 which is a contradiction. because there is no integer bet. 0 & 1.

Hence $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ is an ~~integ~~ irrational no.

2^{n-1} ... $+ \frac{1}{2^{n-1}}$ $\quad \because \frac{1}{p!} \le \frac{1}{2^{p-1}}$

$\Rightarrow$

Theorem #(i) Every monotonically increasing sequence which is not bounded above diverges to $+\infty$ i.e. diverges properly.

(ii) Every monotonically decreasing sequence which is not bounded below diverges to $-\infty$

Proof #(i) Let $\{a_n\}$ is $\uparrow$ which is no bounded above. Then given any $K > 0$, however large, $\exists$ a +ve integer m such that

$$a_m > K.$$

$\because$ $\{a_n\}$ is $\uparrow$

$$a_n \geq a_m > K \quad \forall n \geq m.$$

$$\Rightarrow a_n > K. \quad \forall n \geq m.$$

Thus for every real no $K > 0$, however large, we have a +ve integer m s.that

$$a_n > K. \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \to \infty} a_n = \infty$$

$\Rightarrow \{a_n\}$ diverges to $+\infty$

(ii) Let $\{a_n\} \downarrow$ which is not bounded below. Then given any $K > 0$, large, $\exists$ a +ve integer m s.t

$$a_m < -K$$

$\because$ $\{a_n\} \downarrow$

$$a_n \leq a_m < -K. \quad \forall n \geq m.$$

$$\Rightarrow a_n < -K \quad \forall n \geq m.$$

$\Rightarrow \lim_{n \to \infty} a_n = -\infty \Rightarrow \{a_n\}$ diverges to $-\infty$

Theorem: Every monotone sequence either converges or diverges OR A monotone sequence is never oscillatory.

Proof let $\{a_n\}$ be a monotone sequence, then either $\{a_n\} \uparrow$ or $\{a_n\} \downarrow$

case I let $\{a_n\} \uparrow$

If $\{a_n\}$ is bounded above, then $\{a_n\}$ converges to lub. If $\{a_n\}$ is not bounded, then it diverges to $+\infty$

case II Let $\{a_n\} \downarrow$.

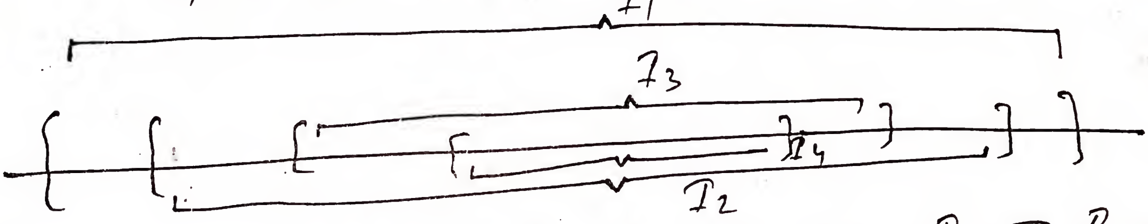
If $\{a_n\}$ is bounded below, then $\{a_n\}$ is cgt to glb. If $\{a_n\}$ is not bounded below, then $\{a_n\}$ is dgt to $-\infty$

Nested Intervals

A sequence $\{I_n = [a_n, b_n]\}$ of closed intervals is called nested if

$$I_{n+1} \subseteq I_n \quad \text{i.e}$$

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \cdots I_n \supseteq I_{n+1} \cdots$$



e.g $I_n = [0, 1/n]$ $\forall n \in N$, Then $I_n \supseteq I_{n+1}$ $\forall n$

In this case 0 belong to all I_n and 0 is the only such common point i.e $\bigcap_{n \in N} I_n = \{0\}$

$+ \frac{1}{2k-1} + \cdots + \frac{1}{2n-1}$ $\quad$ $n,$ $\quad$ $1 - 1/n < 1$

$\in N$ $\quad$ $\therefore \frac{1}{p!} \leq \frac{1}{2^{p-1}}$

In general, a nested interval sequence need not have common point e.g. if $J_n = (0, 1/n)$ $n \in N$, this sequence $\{J_n\}$ is nested but there is no common point because for every $x > 0$ $\exists$ $m \in N$ such that

$$1/m < x \quad \text{(Archimedean)}$$

so that $x \notin J_n$. Similarly the sequence of intervals $K_n = (n, \infty)$ $n \in N$ is nested and have no common point.

However, it is an important property of R that every ~~closed~~ sequence of closed, bounded intervals does have a common point

Nested Interval Property
(Cantor's Nested interval theorem)

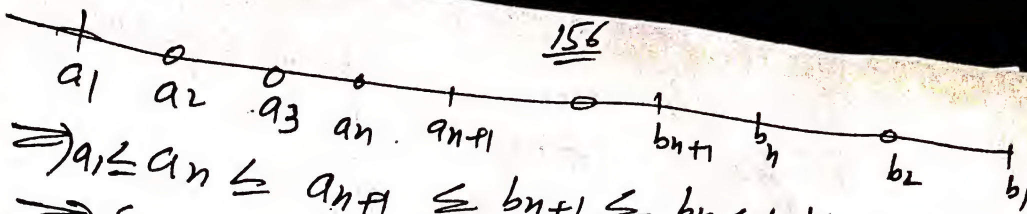
Let $\{I_n = [a_n, b_n]\}$ be a sequence of closed intervals such that

(a) $I_{n+1} \subseteq I_n$ i.e. sequence of nested intervals

(b) $\lim_{n \to \infty} (b_n - a_n) = 0$. Then $\bigcap_{n=1}^{\infty} I_n$ contains unique point i.e. there is exactly one real no common to all intervals I_n.

Proof# Since $I_{n+1} \subseteq I_n$

$\Rightarrow$ $[a_{n+1}, b_{n+1}] \subseteq [a_n \; b_n]$ $\forall n \in N$



$\Rightarrow a_1 \le a_n \le a_{n+1} \le b_{n+1} \le b_n \le b_1 \ \forall n \in N.$

$\Rightarrow \{a_n\}$ is bounded monotone increasing and $\{b_n\}$ is bounded monotone decreasing sequence.

Thus $\{a_n\}$ & $\{b_n\}$ converge.

Let $\lim\limits_{n\to\infty} a_n = a$ (lub) $\lim\limits_{n\to\infty} b_n = b$ (glb)

$\because \ b_n = (b_n - a_n) + a_n$

$\Rightarrow \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}(b_n - a_n) + \lim\limits_{n\to\infty} a_n$

$b = 0 + a \qquad [\because \lim\limits_{n\to\infty}(b_n - a_n) = 0]$

$b = a = x$ (say)

Now x is lub of $\{a_n\}$ &

$a_n \le x \quad \forall n$

Also x is glb of $\{b_n\}$

$x \le b_n \quad \forall n.$

$\Rightarrow \quad a_n \le x \le b_n \quad \forall n.$

$\Rightarrow \quad x \in [a_n, b_n] \quad \forall n$

$\Rightarrow \quad x \in I_n \quad \forall n.$

$\Rightarrow \quad x \in \bigcap\limits_{n=1}^{\infty} I_n$

$\Rightarrow \exists$ a number common to all intervals.

<u>Uniqueness</u> If possible let x, y be two distinct numbers common to all the intervals.

Then $x \in [a_n, b_n] \ \forall n \quad y \in [a_n, b_n] \ \forall n$

If $x < y$, then $a_n \le x < y \le b_n \quad \forall n$

$\Rightarrow\ y-x \leq b_n - a_n \quad \forall n.$

Let $\epsilon = y-x > 0$, then

$$b_n - a_n \geq \epsilon \quad \forall n. \longrightarrow ①$$

Also $\lim_{n\to\infty} (b_n - a_n) = 0$

$\Rightarrow$ for $\epsilon = y-x$ there must exist an integer m such that

$$|(b_n - a_n) - 0| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow\ b_n - a_n < \epsilon \quad \forall n \geq m$$

which contradicts 1. Hence x is only element common to all intervals.

Note # The word closed in above theorem can not be dropped i.e the intersection of ~~all~~ a decreasing sequence of open intervals may be empty.

e.g $I_n = (0, 1/n)\ \forall n \in N$, then $\bigcap_{n\geq 1}^{\infty} I_n = \phi$

Nested Interval Property

If $I_n = [a_n, b_n]$, $n \in N$ is a nested sequence of closed bounded intervals, then $\exists$ a no $x_0 \in R$ such that $x_0 \in I_n\ \forall n \in N$.

Proof $\because$ Intervals are nested, we have

$$I_n \subseteq I_1 \quad \forall n \in N.$$

So that $a_n \leq b_1\ \forall n$ & $a_1 < b_n\ \forall n$

Hence the non-empty set $\{a_n : n \in N\}$ is bounded above and let x_0 be lub of this set

Then $a_n \leq x_0$ ~~$\leq$~~ $\forall n$

, finitely many peak points (peaks) or infinitely

We claim that $x_0 \leq b_n \ \forall n$

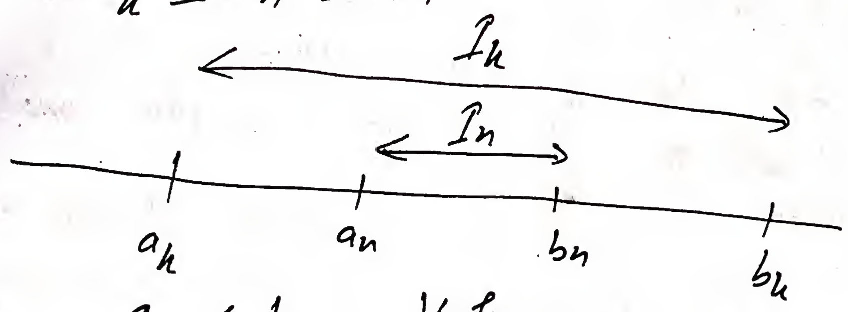
Because for any particular n we show that b_n is an upper bound of $\{a_k : k \in N\}$.

If $n \leq k$, then $I_n \supseteq I_k$, we have.

$$a_k \leq b_k \leq b_n$$

If $k < n$, then since $I_k \supseteq I_n$, we have.

$$a_k \leq a_n \leq b_n$$



Thus $a_k \leq b_n \quad \forall k$

$\Rightarrow$ b_n is an upper bound of the set $\{a_k : k \in N\}$

Hence $x_0 \leq b_n$ for each $n \in N$

$\Rightarrow a_n \leq x_0 \leq b_n \quad \forall n$

$\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} I_n$

Note x_0 above may not be unique.

Peak Point and Peak of a Sequence.

A natural no m is called a peak point of the sequence $\{a_n\}$ if

$$a_n \leq a_m \quad \forall n > m$$

and the term (a_m) is called a peak of sequence. i.e a_m is never exceeded by any term that follows it in the sequence

Note (a) in a decreasing sequence every term is a peak and every natural no is a peak point.
(b) In an increasing sequence no term is peak and no natural no is a peak point

e.g $a_n = \frac{1}{n}$ when $n \leq 5$

Then 1, 2, 3, 4, 5 are five peak points

$a_n = -n$ when $n > 5$

1, 2, 3, 4, 5 are five peak points

(ii) If $a_n = 1$ when $n = 1, 2, \ldots m$

$= -1$ when $n > m$

Then m is only peak point.

(iii) If $a_n = \frac{1}{n}$, then every natural no is a peak point

because for any natural no m, there for $n > m$

$\frac{1}{n} < \frac{1}{m}$ i.e $a_n < m$ $\forall n > m$

Thus a sequence may have no peak point, a finite no of peak points or an infinite no of peak points.

Monotone Subsequence Theorem

Theorem # Every sequence of real nos contain a monotone subsequence.

Proof # Let $\{a_n\}$ be any sequence. Sequence $\{a_n\}$ may have no peak point (peak), finitely many peak points (peaks) or infinitely many

...k points.

...I The sequence has no peak point and ...ce no peak.

...ice 1 is not a peak point, $\exists$ a natural $n_2 > 1$ such that $a_{n_2} > a_1$

... n_2 is not a peak point, $\exists$ a natural no $n_3 > n_2$ s.that $a_{n_3} > a_{n_2}$

Repeating the above argument, we get a ...b-sequence $\{a_{n_k}\}$ such that $n_1 = 1$

$a_{n_1} < a_{n_2} < a_{n_3} < \cdots\cdots$

...s the sequence $\{a_n\}$ contains a monotoni... increasing subsequence.

II # The sequence $\{a_n\}$ has a finite no peak points.

Let m be the largest peak point ... x_m is a peak. Let n_1 be a natural no s.that ...m, then n_1 is not a peak point

$\exists$ a natural no $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$

...ain n_2 is not a peak point

$\exists$ a natural no $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$

...ting the above argument we get a ...sequence $\{a_{n_k}\}$ such that

$a_{n_1} < a_{n_2} < a_{n_3} < \cdots\cdots$

$\{a_n\}$ contains a monotone increasing ...sequence.

... The sequence $\{a_n\}$ has an infinite no ...eak points.

... term (a_m) is called a peak ...

... i.e a_m is never exceeded by any ...that follows it in the sequence

peak points.

Case I The sequence has no peak point and hence no peak.

Since 1 is not a peak point, $\exists$ a natural no $n_2 > 1$ such that $a_{n_2} > a_1$

$\because$ n_2 is not a peak point, $\exists$ a natural no $n_3 > n_2$ s.that $a_{n_3} > a_{n_2}$

Repeating the above argument, we get a sub-sequence $\{a_{n_k}\}$ such that

$$a_{n_1} < a_{n_2} < a_{n_3} < \text{------} \qquad n_1 = 1$$

Thus the sequence $\{a_n\}$ contains a monotonically increasing subsequence.

Case II # The sequence $\{a_n\}$ has a finite no of peak points.

Let m be the largest peak point and x_m is a peak. Let n_1 be a natural no s.that $n_1 > m$, then n_1 is not a peak point

$\therefore$ $\exists$ a natural no $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$

Again n_2 is not a peak point

$\therefore$ $\exists$ a natural no $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$

Repeating the above argument we get a sub-sequence $\{a_{n_k}\}$ such that

$$a_{n_1} < a_{n_2} < a_{n_3} < \ldots.$$

Thus $\{a_n\}$ contains a monotone increasing sub-sequence.

Case III The sequence $\{a_n\}$ has an infinite no of peak points.

and the term (a_m) is called

sequence. i.e a_m is never exceeded by any

term that follows it in the sequence

Let the peak points be $n_1, n_2, n_3 \dots$
such that $n_1 < n_2 < n_3 \dots$

$\because$ n_1 is ~~not~~ a peak point and $n_2 > n_1$

$\therefore$ $a_{n_2} < a_{n_1}$

$\because$ n_2 is not a peak point & $n_3 > n_2$

$\therefore$ $a_{n_3} < a_{n_2}$

Repeating the above process we get a sub-sequence $\{a_{n_k}\}$ so that $a_{n_1} > a_{n_2} > a_{n_3} > \dots$
Thus the sequence $\{a_n\}$ contains a monotonically decreasing sequence $\{a_{n_k}\}$

Theorem (Bolzano Weirestrass)
Every bounded real sequence has a convergent subsequence

Proof # Let $\{a_n\}$ be bounded sequence.
Then there is a closed interval $I_0 = [a, b]$ such that

$$a_n \in [a, b] = I_0 \quad \forall n$$

Bisecting $I_0 = [a, b]$ into two equal intervals

$$\left[a, \frac{a+b}{2}\right], \left[\frac{a+b}{2}, b\right]$$

one of these intervals must contain a_n for infinite many the integer. Let this interval be

$$I_1 = \left[a, \frac{a+b}{2}\right] = [a_1, b_1] \text{ with length } b_1 - a_1 = \frac{b-a}{2}$$

and $I_0 \supset I_1$

Bisecting $I_1 = [a_1, b_1]$ into two equal intervals we get

$$[a_1, \frac{a_1+b_1}{2}], \; [\frac{a_1+b_1}{2}, b_1]$$

one of these contain infinite terms of the sequence $\{a_n\}$. Let this be.

$$I_2 = [a_1, \frac{a_1+b_1}{2}] \quad \text{with length } \frac{b_1-a_1}{2} = \frac{b_1-a_1}{2^2}$$

and $I_1 \supset I_2$

Continuing this process, we obtain a sequence of nested intervals $I_0, I_1, I_2 \dots$ such that

$$I_0 \supset I_1 \supset I_2 \supset I_3 \supset \dots$$

with $b_k - a_k = \frac{b-a}{2^k} \to 0$ as $n \to \infty$

and each interval contain an infinite no of terms of the sequence.

Choose a +ve integer n_1 such that

$$a_{n_1} \in I_1$$

$\because I_2 = [a_2, b_2]$ contain infinite terms of $\{a_n\}$

$\therefore \exists$ +ve integer $n_2 > n_1$ such that

$$a_{n_2} \in I_2$$

Continuing this process we obtain numbers $a_{n_1}, a_{n_2}, a_{n_3} \dots$ such that

$$a_{n_k} \in [a_k, b_k] \quad k = 1, 2, 3 \dots$$

and $n_1 < n_2 < n_3 < n_4 \dots$

of peak points

... is called ... and the term (a_m) ... is never exceded by an ... sequence i.e. a_m ... it follows it in the sequence

Proof $\because$ $\{a_n\}$ is bounded

$\therefore$ By B.W Theorem there is a convergent sub-sequence $\{a_{n_k}\}$.

Also $a_{n_k} > \alpha \quad \forall\, n_k > n_0$

Hence $\lim\limits_{k \to \infty} a_{n_k} \geq \alpha$

Cantor's Intersection Theorem for real line.

Theorem # If $F = \{F_n\}$ is a countable class/family of non-empty closed and bounded sets such that

$$F_1 \supset F_2 \supset F_3 \supset F_4 \cdots \supset F_n \supset \cdots$$

Then $\bigcap\limits_{n=1}^{\infty} F_n$ is non-empty.

Proof # $\because$ each F_n is a non-empty closed and bounded set

$\therefore$ $\exists$ exists sequences of real nos M_n & m_n belong to F_n such that

$M_n = \operatorname{Sup} F_n \qquad m_n = \operatorname{Inf} F_n$

$\because$ $F_n \supset F_{n+1} \quad \forall n$

$\therefore$ $M_n \geq M_{n+1}$ & $m_n \leq m_{n+1} \quad \forall n \in N$

Thus $\{a_{n_k}\}$ is a subsequence of a sequence $\{a_n\}$

We show that $\{a_{n_k}\}$ is cgt.

The sequence $\{c_k\}$ is monotone and bounded and so is cgt.

Let $l = \lim_{k\to\infty} c_k$.

Similarly the sequence $\{d_k\}$ converges to some no m

$$m - l = \lim_{k\to\infty} d_k - \lim_{k\to\infty} c_k$$

$$= \lim_{k\to\infty} (d_k - c_k) = \lim_{k\to\infty} \frac{d-c}{2^k} = 0$$

$m = l$

$\because\ a_{n_k} \in [c_k, d_k] \quad k = 1, 2, 3 \cdots$

$$c_k \leq a_{n_k} \leq d_k.$$

By squeeze theorem

$$\lim_{k\to\infty} a_{n_k} = l = m$$

Thus $\{a_n\}$ contains a convergent subsequence

<u>Corollary</u> Suppose that $\{a_n\}$ is bounded sequence such that $a_n > \alpha \quad \forall\ n > n_0$ where α is a real number, then there is a convergent subsequence $\{a_{n_k}\}$ such that

$$\alpha \leq \lim_{k\to\infty} a_{n_k}$$

ence ...

that follows it in the sequence

Now the lowest bound for $\bigcap_{n=1}^{\infty} F_n$ is the lower bound of the sequence $\{M_n\}$ of upper boundes. Thus $\{M_n\}$ is non-increasing sequence which is bounded below and is therefore convergent.

Let $\lim_{n\to\infty} M_n = M$.

We show that $M \in \bigcap_{n=1}^{\infty} F_n$

Let $M \notin \bigcap_{n=1}^{\infty} F_n$

Then there will be at least one neighhood say $]M-\epsilon, M+\epsilon[$ $\epsilon > 0$ which contains no point of $\bigcap_{n=1}^{\infty} F_n$

$\Rightarrow$ $]M-\epsilon, M+\epsilon[$ contains no point of F_n for some value of n say m

$\Rightarrow$ $]M-\epsilon, M+\epsilon[$ contains no point of F_n for $n \geq m$.

$\Rightarrow$ $M_n \notin]M-\epsilon, M+\epsilon[$ $\forall n \geq m$

Contracting the fact that $\{M_n\}$ converges to M

Hence $M \in \bigcap_{n=1}^{\infty} F_n$

Cluster Points (Limit points) of a sequence

Frequently valid property # A property of statement $P(n)$ is frequently valid.

for a sequence $\{a_n\}$ if for every natural no m $\exists$ at least one $m_1 > m$ such that $P(n)$ is true.

Note An eventually valid statement is also frequently valid.

Cluster point

A real no c is said to be a cluster point of a sequence $\{a_n\}$ if every nbd of c contains infinitely many terms of the sequence i.e.

$$\forall \epsilon > 0 \quad a_n \in (c-\epsilon, c+\epsilon) \text{ for infinitely many values of } n$$

Note A cluster point of a sequence is called a limit point or a condensation point or an accumulation point or a subsequential limit of the sequence

Difference b/w limit and limit point of a sequence

If $l \in R$ is the limit of a sequence $\{a_n\}$, then for $\epsilon > 0$ $\exists$ $m \in N$ such that

$$|a_n - l| < \epsilon \quad \forall n \geq m$$

$\Rightarrow$ every nbd of l contains all except a finite no of terms of the sequence i.e There are

$$\alpha \leq \lim_{k \to \infty} a_{n_k}$$

that follows ...

only a finite no of terms outside each nbd of l. whereas if l is a limit point of sequence $\{a_n\}$, then every nbd of l contains infinitely many terms of the sequence and there may be infinite terms out the nbd i.e it does not exclude the possibility of an infinite no of terms of sequence lying outside the interval or nbd

Hence limit of a sequence is a limit of the sequence but a limit point of a sequence need not be the limit of the sequence. e.g

$\{(-1)^n\}$ has limit points or cluster point $1, -1$ but has no limit

Note # (1) If $a_n = l$ for infinitely many values of n, then l is a limit point of $\{a_n\}$

(2) If for an $\epsilon > 0$, $(l-\epsilon, l+\epsilon)$ for finitely many values of n, then l can not be a cluster point of $\{a_n\}$

(3) Limit point of a sequence need not be a sequence.

Def 2 A real no l is called a cluster point of a sequence $\{a_n\}$ if given $\epsilon > 0$ and a +ve integer m $\exists$ a no $k > m$ s.t

$$|a_k - l| < \epsilon$$

Thus every nbd of l contains a term of the sequence. This is equivalent to saying that

every nbd of l contains 168 infinitely many terms of the sequence because if $(l-\epsilon, l+\epsilon)$ contains only finite no of terms of $\{a_n\}$ say $a_1, a_2 \ldots a_n$

Let $\delta = \min\{|l-a_1|, |l-a_2| \ldots, |l-a_n|\}$

now $(l-\delta, l+\delta)$ contains no term of the sequence which is a contradiction.

clearly above two definitions are equivalent

Def 3 A real no l is called a cluster point of a sequence $\{a_n\}$ if l is limit of some subsequence of $\{a_n\}$.

Note cluster point is also called a subsequential limit

Limit point of range set & cluster point

We note that limit point of the range set $S = \{a_1, a_2, a_3 \ldots\}$ is automatically a cluster point of $\{a_n\}$. The two notions differ only in that for limit points the nbd is deleted where as for cluster point it is not. The distinction is only introduced to cover the possibility that terms of a sequence may be repeated frequently and the range set may be finite and has no limit point whereas sequence has infinite terms and may have cluster point e.g for sequence $\{(-1)^n\}$ the range set $\{-1, 1\}$ has no limit point but $-1, 1$ are cluster point. Thus if a sequence

term that follows ...

has a limit point l, then l is a cluster point but the converse may not be usually true e.g. $\{(-1)^n\}$ has $+1, -1$ cluster points but has no limit point.

Example 1 0 is a limit point of the sequence $\{\frac{1}{n}\}$

Sol For $\epsilon > 0$ $\exists\, m \in N$ s.that $\frac{1}{m} < \epsilon$

$\therefore$ for $n \geq m$ $0 < \frac{1}{n} \leq \frac{1}{m} < \epsilon$

$$\Rightarrow -\epsilon < 0 < \frac{1}{n} < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \frac{1}{n} \in (-\epsilon, \epsilon) \quad \forall n \geq m$$

$\Rightarrow$ Every nbd of 0 contains infinitely many terms of the sequence $\{1/n\}$

Example 2# The sequence $\{(-1)^n\}$ has two limit points

Example#3# The sequence $\{n\}$ has no cluster point

Theorem# If l is a limit point of the range of a sequence $\{a_n\}$, then l is a limit point of the sequence $\{a_n\}$

Proof# Let S = range of $\{a_n\} = \{a_n : n \in N\}$

$\because$ l is a limit point of S

$\therefore$ Every deleted nbd of S contains infinitely many elements of S which are terms of $\{a_n\}$

$\Rightarrow$ Every nbd of S contains infinitely many terms of $\{a_n\}$

$\Rightarrow$ l is a limit point of the sequence $\{a_n\}$

Note# Converse of the above theorem may not be true

Consider

$$a_n = 1 + (-1)^n = \begin{cases} 0 & \text{when } n \text{ is odd} \\ 2 & \text{when } n \text{ is even.} \end{cases}$$

0, 2 are the limit points of the sequence.
But the range $= \{0, 2\}$ is a finite set and finite set has no limit point

(2) If the terms of the sequence are distinct, then the limit points of the sequence are the limit points of the range set.

Theorem # If a sequence converges to l, then l is the only limit point of the sequence

Proof # The sequence $\{a_n\}$ converges to l

$\Rightarrow$ Given $\epsilon > 0$ $\exists$ +ve integer m such that

$$|a_n - l| < \epsilon \quad \forall n \geq m$$

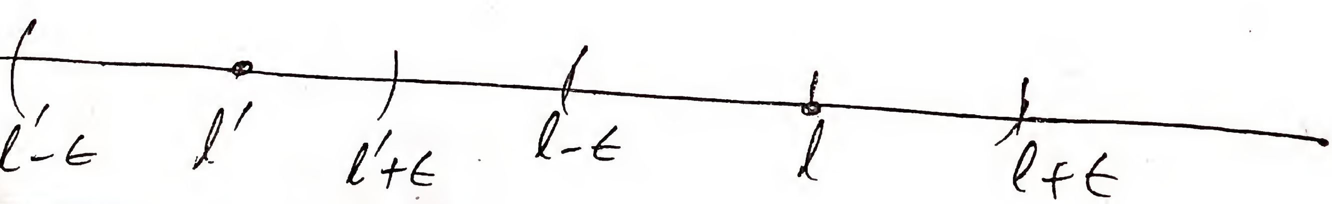
$$\Rightarrow \quad l - \epsilon < a_n < l + \epsilon \quad \forall n \geq m \rightarrow (1)$$

$\Rightarrow$ $a_n \in (l - \epsilon, l + \epsilon)$ for infinitely many values of n.

$\Rightarrow$ every nbd of l contains infinitely many terms of the sequence $\{a_n\}$

$\Rightarrow$ l is a limit point of the sequence $\{a_n\}$

If possible let l' be another limit point of the sequence $\{a_n\}$



Let $\epsilon = \frac{1}{3}(l - l')$ where $l > l'$

Then. $(l'-\epsilon, l'+\epsilon) \cap (l-\epsilon, l+\epsilon) = \phi$

from ① $a_n \in (l-\epsilon, l+\epsilon)$ $\forall n \geq m$

$\therefore$ $a_n \in (l'-\epsilon, l'+\epsilon)$ for almost $m-1$ values of n

$\Rightarrow$ finitely many terms of $\{a_n\}$ lies in $(l'-\epsilon, l'+\epsilon)$

Hence l' is not a limit point of the sequence

Hence l is only limit point of the sequence.

Theorem (Bolzano Weirstrass, theorem.)

Every bounded sequence has at least one limit point

Proof # Let $\{a_n\}$ be a bounded sequence.

and S its range i.e

$$S = \{a_n : n \in N\}$$

$\because$ $\{a_n\}$ is bounded.

$\therefore$ S is bounded.

Case I # Let S be a finite set

The $\exists$ a real no l such that $a_n = l$

for any infinite no of values of $n \in N$.

$\Rightarrow$ Given $\epsilon > 0$, $a_n \in (l-\epsilon, l+\epsilon)$ for an infinite no of values of n

$\Rightarrow$ Every nbd of l contains infinitely many terms of the sequence $\{a_n\}$

$\Rightarrow$ l is a limit point of the sequence $\{a_n\}$

Case II Let S be a infinite set

Since S is an infinite bounded set, by B.W theorem for sets S has at least one limit point say l

Now l is a limit point of S

$\Rightarrow$ Every nbd of l contains an infinite no of elements of S

But each term of S is a term of $\{a_n\}$

$\therefore$ Every nbd of l contains an infinite no of terms of the sequence $\{a_n\}$

$\Rightarrow$ l is a limit point of the sequence $\{a_n\}$

Corollary # If S is a closed and bounded (i.e compact) set, then every sequence in S has a limit point

Proof # A sequence $\{a_n\}$ in S if $a_n \in S \ \forall n$

Let $\{a_n\}$ be a sequence in S, then $a_n \in S \ \forall n$

Since S is bounded, the sequence $\{a_n\}$ is bounded and consequently it has a limit point say l by B.W theorem.

We show that $l \in S$

let $l \in S^c$, then S being closed, S^c is open

$\therefore$ S^c is nbd of l

But S^c contains no term of $\{a_n\}$. This contradicts the fact that l is a limit point of $\{a_n\}$

$\therefore$ $l \notin S^c$. Hence $l \in S$

term that follows

Corollary 2 # If I is a closed interval, then any sequence in I has a limit point in I

Proof # $\because$ I is closed interval

$\therefore$ I is closed and bounded.

$\Rightarrow$ The result follows from Corollary 1.

Theorem # The set of limit points of a bounded sequence is bounded.

Proof let $\{a_n\}$ be a bounded sequence

$\Rightarrow \exists$ real no k & K $(k \le K)$ such that $k \le a_n \le K \quad \forall n \in N$

$\therefore$ $a_n \notin (-\infty, k)$ & $a_n \notin (K, \infty)$ for any n

Let l be any real no

If $l \in (-\infty, k)$, then $(-\infty, k)$ contains no terms of the sequence & l is not limit point of $\{a_n\}$

If $l \in (K, \infty)$, Then (K, ∞) contains no term of sequence $\{a_n\}$ and l is not limit point of $\{a_n\}$

Thus no point outside $[k, K]$ is a limit point of $\{a_n\}$

$\Rightarrow$ The limit point of $\{a_n\}$ lie in $[k, K]$

$\Rightarrow$ The set of all the limit points of a bounded sequence is bounded.

Note The bounds of the set of limit points of a bounded sequence are same as the bounds of sequence

... $\{a_n\}$ i.e. min of all

(2) The set of limit points of an unbounded sequence may or may not be bounded.
e.g $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\}$ is unbounded. but the set of limit points $\{0\}$ is bounded.
The sequence $\{1, 2, 1+\frac{1}{2}, 2+\frac{1}{2}, 2+\frac{1}{3} \dots\}$ is unbounded and the set of limit points is N which is unbounded.

Theorem # Every bounded sequence has the greatest and the least limit points.

Proof # Let $\{a_n\}$ be a bounded sequence.
Then ~~$S = \{a_n$~~ the set E of limit points is also bounded and $E \neq \phi$ (B.W. Theorem)
By Completness property, E has infimum and Supremum.
Let $\inf E = l$ & $\sup E = u$.
For $\epsilon > 0$ let $(u-\epsilon, u+\epsilon)$ be nbd of u.
$\sup E = u \Rightarrow \exists$ some $x \in E$ s.that
$u - \epsilon < x \leq u < u + \epsilon$
$\Rightarrow x \in (u-\epsilon, u+\epsilon)$
$\Rightarrow (u-\epsilon, u+\epsilon)$ is a nbd of x
$\because x \in E$ is a limit point of $\{a_n\}$
$\Rightarrow$ every nbd of x contains infinitely many terms of $\{a_n\}$
$\Rightarrow (u-\epsilon, u+\epsilon)$ contains infinite terms of $\{a_n\}$
This is true for every $\epsilon > 0$
$\Rightarrow u$ is a limit point of $\{a_n\}$
$\Rightarrow u \in E$
Similarly $l \in E$.

Theorem # The set of limit points of a bounded sequence is compact set.

Proof # Let E be the set of all limit points of a bounded sequence $\{a_n\}$.
Then E is closed and bounded subset of R
$\Rightarrow E$ is compact

The Generalised Limits (upper and lower limits)

We have discussed limit of a cgt sequence and have also proved that bounded monotone sequence always converge. There are sequences which are bounded but not monotone. Such sequences can converge but can equally well diverge as

$\{(-1)^{n+1}\}$

From this example we note that a general bounded sequence can diverge by oscillating between various limits. This oscillation suggests trignometric function.

The Lim Sup & Lim Inf are defined for arbitrary (not necessarily cgt) sequences

If $\{a_n\}$ is bounded, then by B.W theorem $\{a_n\}$ has a cgt subsequence. The no $\lim_{n\to\infty} \sup a_n$ is the max value obtainable as the limit of cgt subsequence of $\{a_n\}$ i.e maximum of all limit points/cluster points of a sequence and $\lim_{n\to\infty} \inf a_n$ is the minimum value obtainable as the limit of a cgt subsequence of $\{a_n\}$ i.e min of all

Limit points/Cluster points.

$\lim\limits_{n\to\infty} \operatorname{Sup} a_n$ & $\lim\limits_{n\to\infty} \operatorname{Inf} a_n$ are also denoted by

$$\overline{\lim_{n\to\infty}}\, a_n \quad \& \quad \underline{\lim}\, a_n$$

We discuss these limits under two catergories (a) For bounded sequences (b) For unbounded sequences.

Lim Sup & Lim Inf of Bounded Sequences

Let $\{a_n\}$ be a bounded sequence and E be the set of all limit points of $\{a_n\}$. Then

$$\lim_{n\to\infty} \operatorname{Sup} a_n = \operatorname{lub} E = \operatorname{Sup} E$$

$$\lim_{n\to\infty} \operatorname{Inf} a_n = \operatorname{glb} E = \operatorname{Inf} E$$

Thus $\lim\limits_{n\to\infty} \operatorname{Sup} a_n = u$

if for every +ve ϵ

$|a_n - u| < \epsilon$ for infinitely many values of n and no no larger than u has this property.

$\lim\limits_{n\to\infty} \operatorname{Inf} a_n = v$

if for every $\epsilon > 0$

$|a_n - v| < \epsilon$ for infinitely many values of n & no number less than v has this property.

is true for every $\epsilon > 0$

u is a limit point of $\{a_n\}$

$u \in E$

$l \in E$.

1'7°

Theorem # Let $\{a_n\}$ be a bounded sequence. Then

$$\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n$$

Proof # Let E be the set of subsequential limits of $\{a_n\}$

By definition

$$\inf E \leq \sup E$$

$$\Rightarrow \lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n$$

Theorem # A real no u is the limit superior of a bounded sequence $\{a_n\}$ iff

(a) For each $\epsilon > 0$, $a_n > u - \epsilon$ for infinitely many values of n.

(b) For each $\epsilon > 0$, $a_n < u + \epsilon$ for all except for finitely many values of n.

Proof # Necessity

Let u be Lim Superior of $\{a_n\}$ & let $\epsilon > 0$ be given.

$\because$ u is a limit point / cluster point of a_n

$\therefore$ $a_n \in (u-\epsilon, u+\epsilon)$ for infinitely many values of n

In particular $a_n > u - \epsilon$ for infinitely many values of n

Again since u is the greatest limit point, $u + \epsilon$ is not a limit point and therefore $a_n \geq u + \epsilon$ for only finitely many values of n

(if for some $\epsilon > 0$, $a_n \geq u + \epsilon$ for infinite values of n, then $\{a_n\}$ will have limit point $p \geq u + \epsilon$)

$\therefore$ $a_n < u + \epsilon$ for all except finite values of n

Remarks #(1) For a cgt sequence all subsequences converge to same limit but a dgt bounded sequence has many cgt subsequences. Lim sup & Lim Inf give the behaviour of the set of limit points. This behaviour signifies how much the sequence $\{a_n\}$ can rise or fall when n is very large enough. The set of limit points for a cgt sequence is not empty & is also bounded and hence lub, glb

(2) If $\{a_n\}$ is unbounded, $\lim\limits_{n\to\infty} \sup a_n = \infty$

and $\lim\limits_{n\to\infty} \inf a_n = -\infty$

Example (i) For sequence $\{(-1)^n\}$, the only limit points are $-1, 1$. So $E = \{-1, 1\}$

$\lim\limits_{n\to\infty} \inf a_n = -1 \qquad \lim\limits_{n\to\infty} \sup a_n = \sup E = 1$

(ii) For sequence $\{1/n\}$, the only limit point is $\{0\}$. So $\lim\limits_{n\to\infty} \sup a_n = \lim\limits_{n\to\infty} \inf a_n = 0$

(iii) If $a_n = k \quad \forall n$

$\lim \inf a_n = \lim \sup a_n = k$

(iv) If $a_n = \begin{cases} 2 & \text{when } n \text{ is odd} \\ -n & \text{when } n \text{ is even} \end{cases}$

Then 2 is limit point of $\{a_n\}$ which is unbounded below $\therefore$ $\underline{\lim}\, a_n = -\infty \qquad \overline{\lim}\, a_n = 2$

(v) For sequence $a_n = (-1)^n n \quad \forall n \in N$.

$\lim\limits_{n\to\infty} \inf a_n = -\infty \qquad \lim\limits_{n\to\infty} \sup a_n = \infty$

Sufficiency Let u satisfies both conditions.

Given $\epsilon > 0$, $u - \epsilon < a_n$ for infinitely many values of n and $u + \epsilon > a_n$ for all except finitely many values of n

$\Rightarrow \quad u - \epsilon < a_n < u + \epsilon$ for infinitely many values of n

$\Rightarrow$ u is a limit point of $\{a_n\}$

Now we show that no no greater than u can be limit point of $\{a_n\}$.

Let u' be any other ~~limit~~ point greater than u.

Let p, q be two numbers such that

$$u < p < u' < q$$

By 2nd condition, for each $\epsilon > 0$, $a_n < u + \epsilon$ for all except for finite values of n

Choosing $p - u = \epsilon > 0$, we have.

$$a_n < p \text{ for all except finitely many values of } n$$

and therefore (p, q) is a nbd of u' contains a_n for finitely many values of n. $\Rightarrow$ u' is not a limit point of $\{a_n\}$ and u is the greatest limit of $\{a_n\}$

Hence u is limit superior of $\{a_n\}$

Theorem # A real no l is the limit inferior of a bounded sequence $\{a_n\}$ iff the following are true

(i) for each $\epsilon > 0$, $a_n \not< u + \epsilon$ for infinitely many values of n

(ii) for each $\epsilon > 0$, $a_n > l - \epsilon$ ~~$a_n < u + \epsilon$~~ for all except finitely many values of n

Proof Necessity

Let l be Limit inferior of $\{a_n\}$ and $\epsilon > 0$ be given

$\because$ l is Limit inferior of $\{a_n\}$

$\therefore$ $l - \epsilon < a_n < l + \epsilon$ for infinitely many n

In particular $a_n < l + \epsilon$ for infinitely many values of n

Again since l is the least limit point, $l - \epsilon$ is not a limit point and

$a_n \leq l - \epsilon$ for finitely many values of

because if $\epsilon > 0$, $a_n \leq l - \epsilon$ for infinitely many values of n, then, then $\{a_n\}$ will have a limit point $p \leq l - \epsilon$

$\therefore$ $a_n > l - \epsilon$ for all except finitely many values of n

Sufficiency # Let us assume that l satisfies both the conditions

Given $\epsilon > 0$, $a_n < l + \epsilon$ for infinite values of n

and $a_n > l - \epsilon$ for all except finite values of n

$\Rightarrow$ $l - \epsilon < a_n < l + \epsilon$ for infinitely many values of n

$\therefore$ l is a limit point of $\{a_n\}$

We show that no number less than l can be a limit point of $\{a_n\}$

let l' be be any number less than l

term that follows

Let p, q be two members such that

$p < l' < q < l$

By 2nd condition. for each $\epsilon > 0$

$a_n > l - \epsilon$ for all except for finite values of n

$\Rightarrow$ for $\epsilon = l - q > 0$, we have.

$a_n > l - \epsilon = l - (l - q) = q$ for all except finite values of n

$\Rightarrow$ (p, q) is a nbd of ~~a_n~~ l' containing a_n for finitely many values of n

$\Rightarrow$ l' is not a limit point of $\{a_n\}$ so that l is the least limit point of $\{a_n\}$ and hence l is limit inferior of $\{a_n\}$

Theorem # A sequence $\{a_n\}$ converges to l iff

$$\lim_{n\to\infty} \operatorname{Sup} a_n = \lim_{n\to\infty} \operatorname{Inf} a_n = l$$

Proof # Let the sequence $\{a_n\}$ converges to l. Then given $\epsilon > 0$ $\exists$ +ve integer m such that

$$|a_n - l| < \epsilon \quad \forall n \geq m.$$

$$\Rightarrow l - \epsilon < a_n < l + \epsilon \quad \forall n \geq m$$

Since the nbd $(l-\epsilon, l+\epsilon)$ of l contains a_n for infinitely many values of n and since ϵ is arbitrary therefore every nbd of l contains infinitely many terms of the sequence $\{a_n\}$

$\therefore$ l is a limit point of $\{a_n\}$

We show that l is only limit point of $\{a_n\}$

$\Rightarrow \lim_{n\to\infty} a_n = 0 \quad \Rightarrow \{a_n\}$ has only one cluster

Let l' be any number other than l. Two cases arise

(i) $l < l'$ (ii) $l' < l$

Suppose $l < l'$. Let p, q, r be three numbers such that $p < l < q < l' < r$

$\because a_n \to l$

$\therefore$ Every nbd of l contains a_n for all except finitely many values of n. In particular (p, q) for all except finitely many values of n

$\Rightarrow$ The nbd (q, r) of l' contains a_n for almost finite values of n

$\Rightarrow l'$ can not be a limit point of $\{a_n\}$

Similarly $l' < l$, l' is not a limit point of $\{a_n\}$

Thus l is only limit point of $\{a_n\}$

Hence $\lim\limits_{n\to\infty} \sup a_n = \lim\limits_{n\to\infty} \inf a_n$

<u>Converse</u> Let $\lim\limits_{n\to\infty} \sup a_n = \lim\limits_{n\to\infty} \inf a_n$

Let $\epsilon > 0$ be given.

$\because l = \lim\limits_{n\to\infty} \sup a_n$

$\therefore a_n < l + \epsilon$ for all except finite values of n

$\Rightarrow \exists$ a +ve integer m_1 s.t

$a_n < l + \epsilon \quad \forall n \geq m_1$

Again $l = \lim\limits_{n\to\infty} \inf a_n$

$\therefore a_n > l - \epsilon$ for all except finitely many values of n

$\Rightarrow \exists$ a +ve integer m_2 such that

$a_n > l - \epsilon \quad \forall n \geq m_2$

... of Limit Superior and Limit Inferior (for bounded sequence)

Let $\{a_n\}$ be bounded sequence and bounded above by K. Then for each $n \in N$ the set $S_n = \{a_n, a_{n+1}, \dots\}$ is bounded above by K. By L.U.B. axiom S_n has lub M_n i.e

$$M_n = \operatorname{Sup} S_n = \operatorname{Sup}\{a_n, a_{n+1}, a_{n+2} \dots\}$$

clearly

$$M_n \geq M_{n+1} \quad \forall n \in N$$

Thus the sequence $\{M_n\}$ being decreasing sequence either converges or diverges to $-\infty$.

If $\{M_n\}$ is convergent, then $\lim\limits_{n\to\infty} M_n$ is called limit superior of $\{a_n\}$ i.e.

$$\lim_{n\to\infty} \operatorname{Sup} a_n = \lim_{n\to\infty} M_n$$

If $\{a_n\}$ not bounded above we can let $M_n = \infty$ $\Rightarrow \lim \operatorname{Sup} a_n = \infty$

~~If $\{M_n\}$ is divergent, then $\lim\limits_{n\to\infty} \operatorname{Sup} a_n = +\infty$~~

Let $\{a_n\}$ be bounded below by k. Then each n

$$S_n = \{a_n, a_{n+1}, \dots\}$$ is bounded below by k for each n

$\Rightarrow$ S_n has g.l.b $\underline{m_n}$ 184

Clearly $m_n \le m_{n+1} \quad \forall n \in N.$

Thus sequence $\{m_n\}$ being an increasing sequence is either convergent or diverges to $+\infty$

If $\{m_n\}$ is cgt, then

$$\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} m_n = \lim_{n\to\infty} \operatorname{Inf} S_n$$

$$= \lim \inf \{a_n, a_{n+1}, \dots\}$$

If $\{a_n\}$ is not bounded below we define $\lim \inf a_n = -\infty$

<u>Theorem</u> # (a) If a sequence $\{a_n\}$ is such that

$$\overline{\lim_{n\to\infty}} a_n = \underline{\lim}\, a_n = \infty$$

, then $\{a_n\}$ diverges to ∞

(b) If a sequence $\{a_n\}$ is such that

$\overline{\lim}\, a_n = \underline{\lim}\, a_n = -\infty$, then $\{a_n\}$ diverges.

<u>Proof</u> # (a) Let $M_n = \operatorname{Sup}\{a_n, a_{n+1}, a_{n+2} \dots\} \; \forall n$

and $m_n = \operatorname{Inf}\{a_n, a_{n+1}, a_{n+2} \dots\} \; \forall n.$

Then $m_n \le a_n \le M_n \quad \forall n. \longrightarrow (1)$

Since $\lim_{n\to\infty} m_n = \underline{\lim}_{n\to\infty} a_n = \infty$ & $\lim_{n\to\infty} M_n = \overline{\lim}_{n\to\infty} a_n = \infty$

from (1) $\lim_{n\to\infty} a_n = \infty$

The sequence diverges to ∞

(b) $\because \lim_{n\to\infty} m_n = \underline{\lim}_{n\to\infty} a_n = -\infty = \overline{\lim}\, a_n$

$\Rightarrow \lim_{n\to\infty} a_n = -\infty$

$\Rightarrow \{a_n\}$ diverges to $-\infty$

$\Rightarrow$ I ... $a_n > l - \epsilon$ ...

... term that follows ...

Examples

Q 1 Give exaples of sequences having
(i) no cluster point (ii) one cluster point
(iii) Two cluster points (iv) infinitely many cluster points.

Sol (i) sequence $\{n\}$ has no cluster point
(ii) The sequence $\{\frac{1}{n}\}$ has one cluster point
(iii) The sequence $\{(-1)^n\}$ has two cluster points namely $-1, 1$
(iv) The sequence $\{2, 1+\frac{1}{2}, 2+\frac{1}{2}, 1+\frac{1}{3}, 2+\frac{1}{3}, 3+\frac{1}{3}, 1+\frac{1}{4}, 2+\frac{1}{4}, 3+\frac{1}{4} \cdots\}$

has infinitely many cluster points. Every natural no is a cluster point

Q.2 Find cluster points of the sequences defined by nth term
(i) $(-1)^n$ (ii) 5 (iii) $\frac{(-1)^n}{n}$
(iv) n (v) $(-1)^n\left(1+\frac{1}{n}\right)$ (vi) $\left(1+\frac{1}{n}\right)^{n+1}$
(vii) $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}$

Sol (i) $a_n = (-1)^n = \begin{cases} -1 & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even,} \end{cases}$

$\{a_n\}$ two cluster points.

(ii) $a_n = 5$ is Constant sequence converging to 5
$\Rightarrow \{a_n\}$ has only one cluster point

(iii) $a_n = \frac{(-1)^n}{n} = \begin{cases} -1/n & \text{when } n \text{ is odd} \\ 1/n & \text{when } n \text{ is even.} \end{cases}$

$\Rightarrow \lim_{n\to\infty} a_n = 0 \Rightarrow \{a_n\}$ has only one cluster poin

(iv) for any $l \in R$ $(l-\frac{1}{4}, l+\frac{1}{4})$ contains at most one term of the sequence $\{n\}$

$\therefore$ $l \in R$ is not a limit point of $\{a_n\}$

$\Rightarrow$ The sequence $\{n\}$ has no limit point

(v)

$$a_n = (-1)^n\left(1+\frac{1}{n}\right) = \begin{cases} -\left(1+\frac{1}{n}\right) & n \text{ is odd} \\ \left(1+\frac{1}{n}\right) & n \text{ is even} \end{cases}$$

$\Rightarrow$ Sequence $\{a_n\}$ has two cluster points $-1, 1$

(vi) Here $a_n = \left(1+\frac{1}{n}\right)^{n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) = e \times 1 = e$$

$\Rightarrow$ sequence $\{a_n\}$ converges to e and has one cluster point

(vii) $a_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$

The sequence converges to e

$\therefore$ The sequence has only one cluster point

Q# Find the limit superior and limit inferior of each of the following sequences

(i) $\{1, 3, 5, 1, 3, 5, 1, 3, 5 \ldots\}$

(ii) $\{1, 5, 17, 19, 1, 5, 17, 19, 1, 5, 17, 19, \ldots\}$

(iii) $\{a_n\}$ where $a_n = \sin\frac{n\pi}{3}$

(iv) $\{a_n\}$ where $a_n = (-2)^{-n}\left(1+\frac{1}{n}\right)$

(v) $\{a_n\}$ where $a_n = (-10)^n\left(1+\frac{1}{n}\right)^2$

(vi) $\{a_n\}$ where $a_n = (-1)^n\left(1-\frac{1}{n}\right)$

$\Rightarrow$ $\{a_n\}$ diverges to $-\infty$

$\Rightarrow$ $a_n > l - \epsilon$

494 S

WARNING:

187

(vii) $\{a_n\}$ $\quad a_n = \left(1+\frac{1}{n}\right)^{n+1}$

(viii) $\{a_n\}$ $\quad a_n = (-1)^n(2^n+3^n)$

1 4 7

$a_m = 1+3(m-1)$
$= 3m-2$

Sol (i)

$$a_n = \begin{cases} 1 & \text{if } n = 3m-2 \\ 3 & \text{if } n = 3m-1 \quad m \in N \\ 5 & \text{if } n = 3m \end{cases}$$

The set of cluster points of $\{a_n\} = E = \{1,3,5\}$

$$\overline{\lim_{n\to\infty}} a_n = \operatorname{Max}\{1,3,5\} = 5$$

$$\underline{\lim_{n\to\infty}} a_n = \min\{1,3,5\} = 1$$

(ii) The set of cluster points $= E = \{1, 5, 17, 19\}$

$\overline{\lim_{n\to\infty}} a_n = \operatorname{Max} E = 19$ $\quad \underline{\lim_{n\to\infty}} a_n = \operatorname{Min} E = 1$

(iii)

$$a_n = \sin\frac{n\pi}{3} = \begin{cases} 0 & \text{if } n = 3m \\ \frac{\sqrt{3}}{2} & \text{if } n = 6m-5 \text{ or } 6m-4 \\ -\frac{\sqrt{3}}{2} & \text{if } n = 6m-2 \text{ or } 6m-1 \end{cases}$$

where $m \in N$

The set of cluster points of $\{a_n\} = E = \left\{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right\}$

$\Rightarrow \overline{\lim_{n\to\infty}} a_n = \frac{\sqrt{3}}{2}$ $\quad \underline{\lim_{n\to\infty}} a_n = -\frac{\sqrt{3}}{2}$

(iv) Here $\{a_n\}$ Converges to 0

The set of cluster points $= E = \{0\}$

$\Rightarrow \overline{\lim_{n\to\infty}} a_n = 0 = \underline{\lim_{n\to\infty}} a_n$

$\Rightarrow \overline{\lim_{n\to\infty}} a_n = 0 = \underline{\lim_{n\to\infty}} a_n.$

(V) Here $a_n = (-10)^n\left(1+\frac{1}{n}\right)^2 = \begin{cases} -(10)^n\left(1+\frac{1}{2}\right)^2 & n \text{ odd} \\ (10)^n\left(1+\frac{1}{n}\right)^2 & n \text{ even.} \end{cases}$

$\because \lim_{n\to\infty} (10)^n\left(1+\frac{1}{n}\right)^2 = \infty$

$E = \{-\infty, +\infty\}$

$\overline{\lim_{n\to\infty}} a_n = \infty \qquad \underline{\lim_{n\to\infty}} a_n = -\infty$

(Vi) $a_n = (-1)^n\left(1-\frac{1}{n}\right) = \begin{cases} 1-\frac{1}{n} & n \text{ even.} \\ -1+\frac{1}{n} & n \text{ ood.} \end{cases}$

$E = \{1, -1\}$

$\underline{\lim_{n\to\infty}} a_n = -1 \qquad \overline{\lim_{n\to\infty}} a_n = 1$

(vii) $a_n = \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)$

$\lim_{n\to\infty} a_n = e \times 1 = e$

$\underline{\lim_{n\to\infty}} a_n = \overline{\lim_{n\to\infty}} a_n = e$

(viii) $a_n = (-1)^n(2^n+3^n) = \begin{cases} -(2^n+3^n) & n \text{ is odd} \\ (2^n+3^n) & n \text{ even.} \end{cases}$

$E = \{-\infty, \infty\} \Rightarrow \overline{\lim_{n\to\infty}} a_n = \infty \qquad \underline{\lim_{n\to\infty}} a_n = -\infty$

(VI) $\{a_n\}$ where $a_n = (-1)^n\left(1-\frac{1}{n}\right)$

$\Rightarrow$ [illegible] $_{n\to\infty}$

$\Rightarrow \{a_n\}$ diverges to $-\infty$

$\Rightarrow$ I [illegible] $a_n > l - \epsilon$ [illegible]

Theorem# For a sequence $\{a_n\}$ prove that

$$\underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n.$$

Proof# If $\{a_n\}$ is unbounded, then either $\overline{\lim}_{n\to\infty} a_n = \infty$ or $\underline{\lim}_{n\to\infty} a_n = -\infty$ and hence there is nothing to prove.

Let a_n be a bounded sequence.

Let $m_n = glb\{a_n, a_{n+1}, a_{n+2} \dots\}$

$M_n = lub\{a_n, a_{n+1}, a_{n+2} \dots\}$ $\forall n \in N.$

Then $m_n \le M_n.$

$$\Rightarrow \lim_{n\to\infty} m_n \le \lim_{n\to\infty} M_n$$

$$\Rightarrow \underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n$$

Theorem# If $\{a_n\}$ & $\{b_n\}$ are two sequences such that $a_n \le b_n$ $\forall n \in N$, then

(i) $\overline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} b_n.$ (ii) $\underline{\lim}_{n\to\infty} a_n \le \underline{\lim}_{n\to\infty} b_n$

Proof# Let $M_n = lub\{a_n, a_{n+1}, a_{n+2} \dots\}$

$M_n' = lub\{b_n, b_{n+1}, \dots\}$

$m_n = glb\{a_n, a_{n+1} \dots\}$

$m_n' = glb\{b_n, b_{n+1} \dots\}$

$\underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n.$

$\because \quad a_n \le b_n \quad \forall n$

$\therefore \quad M_n \le M_n'$ & $m_n \le m_n' \quad \forall n.$

$\Rightarrow \lim\limits_{n\to\infty} M_n \le \lim\limits_{n\to\infty} M_n'$ & $\lim\limits_{n\to\infty} m_n \le \lim\limits_{n\to\infty} m_n'$

$\Rightarrow \overline{\lim\limits_{n\to\infty}}\, a_n \le \overline{\lim\limits_{n\to\infty}}\, b_n$ & $\underline{\lim\limits_{n\to\infty}}\, a_n \le \underline{\lim\limits_{n\to\infty}}\, b_n$

<u>Theorem</u> # If $\{a_n\}$ & $\{b_n\}$ are bounded sequences, then show that

(i) $\overline{\lim\limits_{n\to\infty}}\,(a_n+b_n) \le \overline{\lim\limits_{n\to\infty}}\, a_n + \overline{\lim\limits_{n\to\infty}}\, b_n$

(ii) $\underline{\lim\limits_{n\to\infty}}\,(a_n+b_n) \ge \underline{\lim\limits_{n\to\infty}}\, a_n + \underline{\lim\limits_{n\to\infty}}\, b_n$

<u>Proof</u> # Let $M_n = \text{Sup}\{a_n, a_{n+1}, a_{n+2} \dots\}$

$M_n' = \text{Sup}\{b_n, b_{n+1}, \dots\}$

$m_n = \text{Inf}\{a_n, a_{n+1} \dots\}$

$m_n' = \text{Inf}\{b_n, b_{n+1}, \dots\}$

$\because \{a_n\}$ & $\{b_n\}$ are bounded.

$\therefore \{a_n+b_n\}$ is bounded.

$$\text{Sup}\{a_n+b_n, a_{n+1}+b_{n+1} \dots\} \le \text{Sup}\{a_n, a_{n+1} \dots\} + \text{Sup}\{b_n, b_{n+1} \dots\}$$

$$= M_n + M_n'$$

$$\text{Inf}\{a_n+b_n, a_{n+1}+b_{n+1} \dots\} \ge \text{Inf}\{a_n, a_{n+1} \dots\} + \text{Inf}\{b_n, b_{n+1} \dots\}$$

(i) $\overline{\lim\limits_{n\to\infty}}\,(a_n+b_n) = \lim\limits_{n\to\infty} \text{Sup}\{a_n+b_n, a_{n+1}+b_{n+1} \dots\}$

$\le \lim\limits_{n\to\infty} (M_n + M_n')$

$$= \lim_{n\to\infty} M_n + \lim_{n\to\infty} M_n'$$

$$= \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n$$

(ii) $\underline{\lim}_{n\to\infty} (a_n + b_n) \geq \lim_{n\to\infty} (m_n + m_n')$

$$\geq \lim_{n\to\infty} m_n + \lim_{n\to\infty} m_n'$$

$$= \underline{\lim}_{n\to\infty}\, a_n + \underline{\lim}_{n\to\infty}\, b_n$$

<u>Note</u> (1) By Combining the above two

$$\underline{\lim}_{n\to\infty}\, a_n + \underline{\lim}_{n\to\infty}\, b_n \leq \underline{\lim}_{n\to\infty} (a_n + b_n) \leq \overline{\lim_{n\to\infty}} (a_n + b_n)$$

$$\leq \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n$$

(2) In certain cases strict inequalities may hold.

e.g

$a_n = (-1)^n$ & $b_n = (-1)^{n+1}$

Then $\overline{\lim}\, a_n = +1$ $\quad \overline{\lim}\, b_n = 1$

$\because \quad a_n + b_n = 0 \quad \forall n$

$$\Rightarrow \overline{\lim_{n\to\infty}} (a_n + b_n) = 0 < \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n$$

& $\underline{\lim}_{n\to\infty} (a_n + b_n) > \underline{\lim}_{n\to\infty}\, a_n + \underline{\lim}_{n\to\infty}\, b_n.$

Theorem # If $\{a_n\}$ is a bounded sequence, then

(i) $\overline{\lim\limits_{n\to\infty}}(-a_n) = -\underline{\lim\limits_{n\to\infty}}\, a_n$

(ii) $\underline{\lim\limits_{n\to\infty}}(-a_n) = -\overline{\lim\limits_{n\to\infty}}\, a_n$

(iii) $\overline{\lim\limits_{n\to\infty}}(\lambda a_n) = \lambda\, \overline{\lim\limits_{n\to\infty}}\, a_n \qquad \lambda > 0$

(iv) $\underline{\lim\limits_{n\to\infty}}(\lambda a_n) = \lambda\, \underline{\lim\limits_{n\to\infty}}\, a_n \qquad \lambda > 0$

(v) $\overline{\lim\limits_{n\to\infty}}(\lambda a_n) = \lambda\, \underline{\lim\limits_{n\to\infty}}\, a_n \qquad \lambda < 0$

Proof # $\because$ $\{a_n\}$ is bounded.

$\therefore$ $\{-a_n\}$ & $\{\lambda a_n\}$ are also bounded

(i) $\overline{\lim\limits_{n\to\infty}}(-a_n) = \lim\limits_{n\to\infty} \sup\{-a_n, -a_{n+1} \cdots\}$

$= \lim\limits_{n\to\infty} -\inf\{a_n, a_{n+1}, \cdots\}$

$= -\underline{\lim\limits_{n\to\infty}}\, a_n$

(ii) $\underline{\lim\limits_{n\to\infty}}(-a_n) = \lim\limits_{n\to\infty} \inf\{-a_n, -a_{n+1}, \cdots\}$

$= \lim\limits_{n\to\infty} -\sup\{a_n, a_{n+1}, \cdots\}$

$= -\lim \sup\{a_n, a_{n+1} \cdots\}$

$= -\overline{\lim\limits_{n\to\infty}}\, a_n$

(iii) $\overline{\lim\limits_{n\to\infty}}(\lambda a_n) = \lim\limits_{n\to\infty} \sup\{\lambda a_n, \lambda a_{n+1} \cdots\}$

$= \lambda \lim\limits_{n\to\infty} \sup\{a_n, a_{n+1} \cdots\}$

$= \lambda\, \overline{\lim\limits_{n\to\infty}}\, a_n$

Try others.

Subsequential Limit

A real no l is called a Subsequential limit of a sequence $\{a_n\}$ if there is a Subsequence of $\{a_n\}$ converging to $\{a_n\}$

A Subsequential limit a sequence $\{a_n\}$ is also called a cluster point or limit point of the sequence.

Theorem # A real no l is called a Subsequential limit of the sequence $\{a_n\}$ iff each neighbourhood $(l-\epsilon, l+\epsilon)$, $\epsilon > 0$ of l contains infinitely many terms of $\{a_n\}$ (ie iff l is a cluster point of a_n)

Proof # Let l be a Subsequential limit of $\{a_n\}$. Then $\exists$ a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ converging to l

given $\epsilon > 0$ $\exists$ +ven integer k_0 such that

$$|a_{n_k} - l| < \epsilon \quad \forall k \geq k_0$$

$$\Rightarrow a_{n_k} \in (l-\epsilon, l+\epsilon) \quad \forall k \geq k_0$$

$\Rightarrow$ Infinitely many terms of sequence $\{a_{n_k}\}$ & hence of $\{a_n\}$ lie in $(l+\epsilon, l-\epsilon)$

Converse Let each nhd of $(l-\epsilon, l+\epsilon)$ of l contains infinitely many terms of $\{a_n\}$

Then $a_n \in (l-\epsilon, l+\epsilon)$ for infinitely many values of n

In particular for $\epsilon = \frac{1}{n}$

$a_n \in (l-\frac{1}{n}, l+\frac{1}{n}) = I_n$ " " "

Choose $a_{n_1} \in (l-1, l+1)$. Then $\exists\ n_2 > n_1$ such that $a_{n_2} \in (l-1/2, l+1/2) = I_2$

Continuing like this $\exists$ a natural no n_{k_0} s. that

$$n_{k_0} > \cdots n_2 > n_1 \text{ \& } a_{n_{k_0}} \in \left(l-\frac{1}{k_0}, l+\frac{1}{k_0}\right) = I_{k_0}$$

Again continuing in this way get a subsequence $\{a_{n_k}\}$ of $\{a_n\}$

Now for all $n_k \geq n_{k_0}$, we have $k \geq k_0$

$$\Rightarrow \frac{1}{k} \leq \frac{1}{k_0} \quad \text{\&} \quad -\frac{1}{k} \geq -\frac{1}{k_0}$$

$$\Rightarrow l+\frac{1}{k} \leq l+\frac{1}{k_0} \text{ \& } l-\frac{1}{k} \geq l-\frac{1}{k_0}$$

$$\Rightarrow \left(l-\frac{1}{k}, l+\frac{1}{k}\right) \subset \left(l-\frac{1}{k_0}, l+\frac{1}{k_0}\right)$$

$$\Rightarrow \quad I_k \subset I_{k_0} \quad \forall\, k \geq k_0 \quad \forall\, n_k \geq n_{k_0}$$

$$\Rightarrow \forall\, n_k \geq n_{k_0},\ a_{n_k} \in I_k \Rightarrow a_{n_k} \in I_{k_0}$$

$$\Rightarrow a_{n_k} \in \left(l-\frac{1}{k_0}, l+\frac{1}{k_0}\right) \quad \forall\, n_k \geq n_{k_0}$$

$$\Rightarrow \quad |a_{n_k} - l| < \frac{1}{k_0} = \epsilon \quad \forall\, n_k \geq n_{k_0}$$

$\Rightarrow \{a_{n_k}\}$ Converges to l

$\Rightarrow l$ is a subsequential limit of the sequence.

By Muhammad Hussain Assistant Professor Govt Asghar Mall College Rawalpindi